Massachusetts Institute of Technology
Department of Economics
Working Paper Series

IDENTIFICATION OF STANDARD AUCTION MODELS

Susan Athey, MIT
Philip A. Haile, Univ of Wisconsin-Madison

Working Paper 00-18
August 2000

Room E52-251
50 Memorial Drive
Cambridge, MA 02142

Massachusetts Institute of Technology
Department of Economics
Working Paper Series

IDENTIFICATION OF STANDARD AUCTION MODELS

Susan Athey, MIT
Philip A. Haile, Univ of Wisconsin-Madison

Working Paper 00-18
August 2000

Room E52-251
50 Memorial Drive
Cambridge, MA 02142

Identification of Standard Auction Models

Susan Athey and Philip A. Haile*

This version: August 8, 2000

Abstract

We present new identification results for models of first-price, second-price, ascending (English), and descending (Dutch) auctions. We analyze a general specification of bidders' preferences and the underlying information structure, nesting as special cases the pure private values and pure common values models, and allowing both ex ante symmetric and asymmetric bidders. We address identification of a series of such models and propose strategies for discriminating between them on the basis of observed data. In the simplest case, the symmetric independent private values model is nonparametrically identified even if only the transaction price from each auction is observed. For more complex models, we provide conditions for identification and testing when additional information of one of the following types is available: (i) one or more bids in addition to the transaction price; (ii) exogenous variation in the number of bidders; (iii) bidder-specific covariates that shift the distribution of valuations; (iv) the ex post realization of the value of the object sold. Our results include new tests that distinguish between private and common values models.

Keywords: auctions, nonparametric identification and testing, private values, common values, asymmetric bidders, unobserved bids, order statistics

---

* M.I.T. and NBER (Athey) and University of Wisconsin-Madison (Haile). We are grateful to Jerry Hausman, Guido Kuersteiner, Whitney Newey, Harry Paarsch, Rob Porter, Susanne Schennach, Gautam Tripathi, and seminar participants at the Clarence Tow Conference on Auctions (Iowa) for helpful discussions. Astrid Dick, Grigory Kosenok, and Steve Schulenberg provided excellent research assistance. Financial support from NSF grants SBR-9631760 and SES-9983820 (Athey) and SBR-9809802 (Haile) is gratefully acknowledged. Web addresses: http://web.mit.edu/athey/www/ and http://www.ssc.wisc.edu/~phaile/.

1 Introduction

This paper derives new results regarding nonparametric identification and testing of models of first-price sealed-bid, second-price sealed-bid, ascending (English), and descending (Dutch) auctions. The theory literature has focused on several alternative models of the economic primitives in these auctions. In private values models, each bidder knows the value he places on winning the object, but not the values of his opponents. In common values models, information about the value of the object is spread among bidders. Within these classes of models, bidders may be symmetric or asymmetric, and information may be independent across bidders or correlated. A variety of policy issues, including market design, the optimal use of reserve prices, and the effect of increased bidder participation on revenues, depend critically on which of these models describes the environment and on the specific distributions characterizing the demand and information structure.

We consider a general specification of bidders' preferences and information, nesting as special cases the pure private values and pure common values models, and allowing correlated private information and bidder asymmetry. We address econometric identification of a series of these models and propose strategies for discriminating between them. Hence, our results address key empirical challenges facing researchers hoping to evaluate the structure of demand at auctions and to use this information to guide the design of markets.

The types of data available from auctions vary by application. In most cases, the transaction price is observed. We consider four additional types of data that might be available: (i) one or more bids in addition to the transaction price; (ii) bids from auctions with exogenously varying numbers of bidders; (iii) bidder-specific covariates that shift valuations; (iv) the realized value of the object for sale or, more generally, auction-specific covariates. In many auctions, losing bids are not recorded—in descending auctions losing bids are not even made. In oral "open outcry" auctions we may lack confidence in the interpretation of losing bids even when they are observed. In contrast, in sealed-bid auctions and ascending "button" auctions (Milgrom and Weber (1982)), the interpretation of losing bids is clear and records of all bids may be available. Exogenous variation in the number of bidders may arise when auctions are held in different locations, when internet auctions are conducted over different lengths of time, or when the seller (e.g., a government agency) restricts entry. It may also arise by design in field experiments (Engelbrecht-Wiggans et al. (1999)). Bidder-specific covariates such as firm size, location, or inventories are often observed, particularly for government auctions, as are auction-specific covariates such as the appraised value or other characteristics of an object for sale. The realized value of the object sold is observed for several important government auctions, including those of offshore mineral leases (Hendricks and Porter (1988)) and timber contracts (Athey and Levin (2000)). In other cases resale prices can

provide measures of realized values (e.g., McAfee, Takacs, and Vincent (1999)).

Most prior research on identification of auction models has focused on first-price auctions with private values, under the assumption that all bids are observed. Theory predicts a one-to-one mapping between observed bids and the latent private values. When all bids are observed, this mapping can be used to infer the distribution of values from that of the bids (Guerre, Perrigne and Vuong (2000)). In second-price and ascending bid auctions the equilibrium mapping from values to bids is the identity function, simplifying the problem. However, while the econometrician may sometimes observe all bids in second-price auctions, ascending auctions end when the next-to-last bidder drops out. Hence, the winning bidder never reveals her value, implying that at best (if exit prices for all losing bidders are observed) the econometrician observes all but the highest value. Omitting even one order statistic creates challenges for nonparametric identification of the distribution of values. Similar but more difficult issues arise in common values models.

Our main results demonstrate how different data configurations, characterized by additional information of types (i)–(iv) described above, can be used for identification and testing of alternative models. Due to the prior attention to first-price auctions (see, e.g., the recent survey by Perrigne and Vuong (1999)), our primary focus is on second-price and ascending auctions. These are the most common auction forms in practice, particularly with the rising popularity of internet auctions (Lucking-Reiley (2000)).[1] However, the ideas we develop for these auctions also enable us to build on recent work on first-price auctions, yielding identification and testable restrictions even when not all bids are observed, including the special case of a descending auction.

Beginning with second-price and ascending auctions, we first show that the symmetric independent private values (IPV) model is identified when only the transaction price is observed. We then consider a richer data configuration, where more than one bid from each auction is observed. This enables testing of the symmetric IPV model and aids identification and testing of the asymmetric IPV model. However, we also establish a negative result for a more general model allowing arbitrary correlation among values: if even one bid is unobserved in each auction (always the case for ascending auctions), this unrestricted private values model is not identified from observed bids.

In spite of this lack of identification, we show that the unrestricted private values model can be *tested* when there is exogenous variation in the number of bidders. Laffont and Vuong (1996) have shown that private and common values models are observationally equivalent under the assumptions that there is no reserve price and the number of bidders is fixed. Exogenous variation in the number

[1] The prevalence of ascending auctions is well known. Second-price auctions have been used to allocate public resources such as radio spectrum (Crandall (1998)). In ascending auctions, the use of agents (human or software) who bid according to pre-specified cutoff prices results in an auction game equivalent to a second-price sealed bid auction. Bajari and Hortaçsu (2000) have argued that due to other features of bidding at online auctions, these are best viewed as second-price auctions.

of bidders, however, provides a promising avenue for testing because the winner's curse (absent in private values auctions) causes a bidder's willingness to pay to decrease in the number of opponents. We use this fact to develop nonparametric tests of private values models that can be used when two or more bids are observed from each auction.

Next, we consider a potential remedy for the non-identification result for the unrestricted private values model, relying on bidder-specific covariates. We build on the literatures on competing risks (Heckman and Honoré (1989), Han and Hausman (1990)) and the Roy model (Heckman and Honoré (1990)). These literatures consider situations in which the maximal or minimal realization from a random sample is observed; they show that covariates with sufficient variation can be used to recover the underlying joint distribution of interest. These results cannot be applied directly to second-price and ascending auctions when only the transaction price is observed because this price reveals the *second-highest* realization rather than the highest. However, we show that these results can be generalized, implying that bidder-specific covariates can give nonparametric identification in asymmetric private values models (without restriction on the correlation among bidder values) even if only the transaction price from each auction is observed.

Our fourth approach is to use ex post information about the value of the object sold. This idea has previously been exploited by Hendricks, Pinkse and Porter (1999) in first-price sealed-bid auctions.[2] We show that ex post value information (or, more generally, auction-specific covariates) can be used for identification and testing outside the pure common values model they study, and in cases where some bids are unobserved.

Finally, we consider first-price auctions in which some bids are unobserved. Guerre, Perrigne, and Vuong (1995) have shown that the symmetric IPV model is identified from observation of the transaction price alone in first-price auctions; however, existing results for more general models have relied on observation of all bids. We show, for example, that the asymmetric IPV model is identified from the transaction price and identity of the winner. Indeed, all our identification and testing approaches based on observation of one bid or two bids from each auction extend to first-price auctions.

Our work contributes to a growing literature on structural estimation of auction models.[3] Most of this literature has focused on the IPV paradigm. Paarsch (1992b), Laffont and Vuong (1993), Laffont, Ossard and Vuong (1995), Donald and Paarsch (1996), Baldwin, Marshall, and Richard (1997), Deltas and Chakraborty (1997), Bajari (1998), Donald, Paarsch and Robert (1999), Hong

[2]See also Hendricks and Porter (1988) and Athey and Levin (2000), who test implications of common values models using ex post values.

[3] See Hendricks and Paarsch (1995), Laffont (1997), Perrigne and Vuong (1999) and Hendricks and Porter (2000) for surveys of the recent empirical literature on auctions.

and Shum (1999), and Haile (2000) have all proposed estimation techniques relying on parametric distributional assumptions for identification. Nonparametric approaches have been proposed by Guerre, Perrigne, and Vuong (2000) for first-price auctions and by Haile and Tamer (2000) for ascending auctions. Only a few papers, including Paarsch (1992a), Li, Perrigne and Vuong (1998, 1999), Hendricks, Pinkse and Porter (1999), Bajari and Hortaçsu (2000) have considered structural estimation outside the IPV paradigm. Each of these either relies on parametric distributional assumptions or addresses only first-price auctions where all bids are observed. Other authors have proposed tests for the winner's curse based on variation in the number of bidders (e.g., Paarsch (1992a), Haile, Hong, and Shum (2000), and Bajari and Hortaçsu (2000)), although these either require parametric distributional assumptions or observation of all bids. No prior work has considered nonparametric identification and testing of the standard alternative models of ascending and second-price sealed-bid auctions. To our knowledge, the only prior work addressing nonparametric identification for cases in which some bids are unobserved applies to symmetric IPV first-price auctions (Guerre, Perrigne, and Vuong, 1995). Nonparametric tests of alternative valuation and information structures have been considered only for only a few cases.[4]

The remainder of the paper is organized as follows. We first describe our general structural framework and review the equilibrium predictions of the theory of ascending and second-price auctions. In Section 3 we consider identification and testing of private values models. Section 4 then takes up the case of common values. Section 5 extends many of our results to first-price auctions in which some bids are unobserved. Section 6 discusses the robustness of our results to bidder uncertainty regarding the number of opponents they face and to the seller's use of a reserve price. We conclude in Section 7.

2 The Model

Our model is based on that of Milgrom and Weber (1982). We consider an auction of a single indivisible unit, with $n \geq 2$ risk-neutral bidders. In the basic model, the number of bidders n is common knowledge and there is no reserve price. Each bidder $i = 1, \ldots, n$ would receive utility[5]

$$U_i = V + A_i$$

[4] Other tests for common values are considered in Hendricks, Pinkse and Porter (1999), which requires binding reserve prices, and Haile and Tamer (2000), which proposes a test of the IPV assumption for English auctions.

[5] Below we will generalize this structure to allow U_i to depend on auction- and/or bidder-specific observables.

from winning the object, where the random variables $\mathbf{A} = (A_1, \ldots, A_n) \in \mathbb{R}^n$ and $V \in \mathbb{R}$ are drawn from a joint distribution $F_{\mathbf{A},V}(\cdot)$.[6] Let $F_V(\cdot)$, $F_{\mathbf{A}}(\cdot)$, and $F_{A_i}(\cdot)$ denote the marginal distributions of V, $\mathbf{A}$, and A_i, respectively. The distribution of $\mathbf{A}$ conditional on $V = v$ is denoted $F_{\mathbf{A}}(\cdot|v)$. For convenience we let $F_{\mathbf{U}}(\cdot)$ and $F_{U_i}(\cdot)$ denote the induced distributions on $\mathbf{U} = (U_1, \ldots, U_n)$ and U_i.

Each bidder i's private information consists of a signal X_i that is affiliated with U_i. Define $\mathbf{X} = (X_1, \ldots, X_n)$, and let $F_{\mathbf{X}}(\cdot)$ be the joint distribution of $\mathbf{X}$. For most of the paper we assume that the joint distribution of $(\mathbf{A}, V, \mathbf{X})$ is exchangeable with respect to the bidder indices, implying that the joint distributions $F_{\mathbf{A}}(\cdot)$, $F_{\mathbf{U}}(\cdot)$, and $F_{\mathbf{X}}(\cdot)$ are exchangeable, in which case $F_{A_i}(\cdot) = F_A(\cdot)$ for all i, etc. When we relax the exchangeability assumption, we will explicitly refer to the model as *asymmetric*. For a sample of generic random variables $S = (S_1, \ldots, S_n)$ drawn from a distribution $F_{\mathbf{S}}(\cdot)$, we denote by $S^{(j:n)}$ the j^{th} order statistic, with $S^{(n:n)}$ the largest value by convention. Similarly, $F_S^{(j:n)}(\cdot)$ denotes the marginal distribution of $S^{(j:n)}$.

Although bidder-specific random variables enter the bidder's utility additively in our model, the multiplicatively separable specification $U_i = V \cdot A_i$ (see, e.g., Wilson (1998) or Li, Perrigne and Vuong (1999)) can be incorporated in this framework by taking logarithms. Indeed, a natural interpretation of the model is that $\mathbf{U}$, V, $\mathbf{A}$, and $\mathbf{X}$ are in logarithms, with bids in logarithms as well. To simplify the exposition, we will discuss the variables without reference to the logarithmic scale and will allow the random variables (as well as bids) to take on negative values.[7]

Our model nests a wide range of specifications of bidders' preferences and the underlying information structure. There are two main categories of models:

Private Values (PV): $X_i \equiv U_i \;\; \forall i$.[8]

Common Values (CV): $(\mathbf{U}, \mathbf{X})$ are affiliated, and for all i, j, (U_i, X_j) are strictly affiliated but not perfectly correlated.

We will consider only models that fall into one of these classes. In the private values model, no bidder has private information relevant to another's expected utility.[9] In contrast, in the common values model, bidder j would update her beliefs about her utility, U_j, if she observed X_i in addition to her own signal X_j. Thus, there is a "winner's curse" in the common values model: upon learning

[6] When the distribution of $\mathbf{A}$ is unrestricted, the random variable V can be normalized to zero without loss of generality; we include it in our notation to simplify the exposition of several special cases of the model.

[7] In some cases, we will assume that random variables are affiliated; recall that this condition is preserved by monotonic transformations such as the logarithm.

[8] $V \equiv 0$ with all X_i i.i.d. is also a (symmetric) private values model, although here the two private values models are observationally equivalent.

[9] Following the literature, in private values models we will use the terms "utility;" "valuation," and "value" interchangeably.

that she has won, bidder j realizes that her beliefs were more optimistic than her opponents', and her beliefs about her own utility become more pessimistic. The CV model is a special case of the "general affiliated values" model of Milgrom and Weber (1982), where we have restricted the model to rule out pure private values and guarantee that the winner's curse arises. In general, the CV model allows for utilities to differ across bidders; in contrast, in the "pure common values model," the value of the object is the same for all bidders. Several of our results concern this special case.

Pure Common Values (Pure CV): In addition to the CV assumptions, $A_i \equiv 0$, so that $U_i = V\,\forall i$.

Further refinements of the pure CV model will be discussed in Section 4. Now return to the PV model. We have not assumed affiliation, since many of our results hold without this restriction. However, we will pay particular attention to three special cases of an affiliated private values model:

Conditionally Independent Private Values (CIPV): $X_i = U_i = V + A_i\ \forall i$, with $(A_1, \ldots, A_n)$ independent conditional on V.[10]

CIPV with Independent Components (CIPV-I): $X_i = U_i = V + A_i\ \forall i$, with $(A_1, \ldots, A_n, V)$ mutually independent.

Independent Private Values (IPV): $V \equiv 0$, so that $X_i = U_i = A_i\,\forall i$, with $(A_1, \ldots, A_n)$ mutually independent.

We assume that the underlying value and signal distributions, and the number of bidders n, are common knowledge among the bidders. We initially consider second-price sealed-bid and ascending (English) auctions.[11] In a second-price sealed-bid auction bidders submit sealed bids, with the object going to the high bidder at a price equal to the second-highest bid. For ascending auctions we assume the standard "button auction" model of Milgrom and Weber (1982), where bidders exit observably and irreversibly as the price rises exogenously until only one bidder remains.[12] We let the random variable B_i denote the bid made by player i, with $H_{B_i}(\cdot)$ its distribution.

[10] The variable V can be thought of as the random mean of the private values. In practice this can arise from unobserved (to the econometrician) heterogeneity in the objects for sale. Equivalently, each U_i could be the sum of a private component A_i and an independent common component ν about which bidders have identical expectations: $E[\nu] = v\ \forall i$. Below we generalize the CIPV and CIPV-I models to allow private values that are independent conditional on a *vector* of auction characteristics.

[11] We use the terms English auction and ascending auction interchangeably. For a review of standard auction forms, see Milgrom and Weber (1982), McAfee and McMillan (1987a), Milgrom (1985), or Wilson (1992).

[12] This is a stylized model of an ascending bid auction that may match actual practice better in some applications than others. Ascending auctions with "activity rules" (e.g., the FCC spectrum auctions discussed in McAfee and McMillan (1996)), for example, are often designed specifically to replicate the button auction. Many of our results rely only on the interpretation of the transaction price from an ascending auction as the realization of the second-highest of the bids prescribed by the Milgrom-Weber equilibrium.

In the second-price auction, each bidder i's equilibrium bid when $X_i = x_i$ is[13]

$$b_i = b(x_i) = E[U_i | X_i = \max_{j \neq i} X_j = x_i]. \tag{1}$$

In the PV model this reduces to $b_i = u_i = x_i$. In the CV model, strict affiliation of (U_i, X_j) implies that the bid function $b(\cdot)$ is strictly increasing.

Equilibrium strategies are similar in the English auction, although bidders condition on the signals of opponents who have already dropped out, since these are inferred from their exit prices. Letting L_i denote the set of bidders with signals below x_i, this implies an equilibrium exit price

$$b_i = E[U_i | X_i = X_j = x_i \forall j \notin \{i \cup L_i\}, X_k = x_k \forall k \in L_i] \tag{2}$$

for each bidder i. In the case of private values, this again reduces to $b_i = u_i = x_i$. Note that the ascending auction ends at the price $b^{(n-1:n)}$.

2.1 Data Configurations

We consider data sets consisting of observations from T independent auctions. In the basic model, the same group of n bidders participates in each of the T auctions. The joint distribution of $(\mathbf{A}, V, \mathbf{X})$ (conditional on auction-specific covariates, if any) is fixed across auctions. Each auction represents an independent draw from this distribution.[14]

We will assume that the following are observed by the econometrician:

- The transaction price. This is $B^{(n-1:n)}$ in a second-price or ascending auction, but $B^{(n:n)}$ in a first-price or descending auction.
- The number of bidders, n.

In addition, the following data elements may or may not be observed:

- Other bids (or exit prices), in addition to the transaction price, of the form $B^{(m:n)}$.
- The identities of some bidders, where $I^{(m:n)}$ denotes the bidder bidding $B^{(m:n)}$.

[13] Throughout the paper we focus on symmetric equilibrium when bidders are ex ante symmetric. When we allow bidder asymmetry in the private values framework we will focus on the unique equilibrium in weakly undominated strategies, in which each bidder bids her valuation.

[14] Such a dataset might arise as a result of non-cooperative bidding in auctions for procurement contracts or natural resources, where the underlying competitive environment is stationary over the relevant time period, and the contracts are small from the perspective of the bidders. In some applications we might expect dependence of bidders' information and/or willingness to pay on outcomes of prior auctions. Examining the empirical implications of such dependence is a valuable direction for future research. Here we follow the vast majority of the theoretical and empirical literature by focusing on cases in which dependence across auctions is absent.

- K sets of auctions, each with a different number of bidders, $n \in \{n_1, \ldots, n_K\}$ (K and each n_K finite), where the variation in n is exogenous.

- The *ex post* realization of V or, more generally, auction-specific covariates W_0 affecting the distribution of $\mathbf{U}$.

- Bidder-specific covariates, $\mathbf{W} = (W_1, \ldots, W_n)$, where bidder i's utility is given by $U_i = V + A_i + g_i(W_i)$ and $g_i(\cdot)$ is an unknown function.

The possibility that the number of bidders might vary exogenously merits comment. This could arise if (outside the formal model described above) there is a pool of potential bidders who receive random shocks to the cost of participating, which are independent of U_i and X_i. Bidders with positive shocks participate and learn X_i. There is no reserve price, so all bidders who learn X_i place a bid in the auction. This would lead to exogenous variation in the number of bidders. As noted already, exogenous variation in n can also arise from participation restrictions by the seller (e.g., in government auctions or in field experiments) or as the result of varying online auction lengths, where longer auctions may enable more potential bidders to become aware of the sale.

When considering variable numbers of bidders, we may wish to revisit the hypothesis that the number of bidders is common knowledge to the participants. Unlike first-price auctions, in private-value second-price and ascending auctions, bidders' strategies do not depend on the number of opponents, so it is not important that the number of bidders be common knowledge among bidders. In contrast, in the CV model, the assumption plays a more important role. In Section 6.1, we extend the model to the case in which bidders are uncertain how many competitors they face.

Our focus is on identification. However, when we discuss estimation, the relevant asymptotics are taken as the number of auctions approaches infinity. When we consider auctions with a varying number of bidders, $n \in \{n_1, \ldots, n_K\}$, we consider asymptotics as the number of auctions in each of the K sets goes to infinity.

3 Ascending and Second-Price Auctions with Private Values

If all bids are observed in a second-price sealed-bid private values auction, the distribution of values is equal to the distribution of bids. However, if any bids are unobserved (always the case in an ascending auction), the problem becomes more complicated. We proceed by considering identification in a series of nested models of private values auctions, showing how richer models require richer data configurations for identification and testing. As discussed above, we focus in this section and the next on ascending and second-price auctions, and these auction forms are assumed unless otherwise stated.

3.1 Symmetric Independent Private Values

We begin with the symmetric IPV model. In this model, the underlying distribution of valuations is nonparametrically identified even when only one bid per auction is observed. The model can be tested if more than one bid is observed or there is variation in the number of bidders.

Proposition 1 *(i) In the IPV model, $F_{\mathbf{U}}(\cdot)$ is identified from the transaction price. (ii) The IPV model is testable if either (a) more than one bid per auction is .observed or (b) transaction prices at auctions with different numbers of bidders are observed.*

Proof. (i) The i^{th} order statistic from an i.i.d. sample of size n from an arbitrary distribution $F(\cdot)$ has distribution

$$F^{(i:n)}(z) = \frac{n!}{(n-i)!(i-1)!} \int_0^{F(z)} t^{i-1}(1-t)^{n-i}dt \tag{3}$$

(see for example Arnold et. al (1992, p. 13)). Because the right-hand side of (3) is strictly increasing in $F(z)$, the marginal distribution of U_i, $F_U(\cdot)$, is identified whenever any distribution $F_U^{(i:n)}(\cdot)$ is. Since the observed transaction price is equal to the order statistic $U^{(n-1:n)}$ and the marginal distribution $F_U(\cdot)$ completely determines $F_{\mathbf{U}}(\cdot)$, the result follows.

(ii) Under the IPV assumption, values of $F_U(u)$ implied by the distributions of different order statistics ($U^{(n-i:n)}$ and either $U^{(n-j:n)}$ or $U^{(\hat{n}-i:\hat{n})}$ with $i \neq j, \hat{n} \neq n$) must be equal for all u. □

3.1.1 Issues for Estimation and Testing

Although a full analysis of estimation and testing is beyond the scope of this paper, we do offer several observations. First, note that (3) suggests an estimation method. The empirical distribution function $\hat{F}_U^{(i:n)}(\cdot)$ is a pointwise-consistent estimator of $F_U^{(i:n)}(\cdot)$. Because the right side of (3) is differentiable with respect to $F(z)$, the implicit function theorem and continuous mapping theorem imply that the value of $F(z)$ solving (3) when $\hat{F}_U^{(i:n)}(z)$ is substituted for $F^{(i:n)}(z)$ is a consistent estimator of $F_U(z)$.[15]

The testing principle suggested in part (ii) of Proposition 1 can be implemented in a number of ways. A direct Kolmogorov-Smirnov test for equality of distributions can be implemented using bootstrap critical values (see, for example, Romano (1988)). Alternative possibilities include standard tests of equality of means or quantiles of the distributions. The same approaches can be taken for a number of the tests we propose below.

[15] If more than one bid is observed in each auction, more efficient estimates can be obtained by incorporating all observed bids using (3).

In real-world ascending auctions, the button auction rules may not be used; instead, bidding is often free-form. Thus, some caution is warranted when implementing this test using different bids from the same auctions.[16] The IPV button-auction model may be rejected because recorded bids understate the true willingness-to-pay of the bidders. In that case, one could compare estimates obtained from low-ranked bids to those obtained from high-ranked bids to assess the magnitude of the bias that arises from bidders' failing to bid at prices as high as their true valuations.

3.2 Asymmetric Independent Private Values

If the exchangeability assumption is relaxed, we obtain a model of asymmetric independent private values: bidders' valuations are independent, but some bidders may be more likely to have high valuations than others. Specifically, each U_i is drawn from an idiosyncratic distribution $F_{U_i}(\cdot)$. While the previous result relied heavily on valuations' being identically distributed across bidders, identification in the case of asymmetric private values can still be obtained from observation of a single bid at each auction, as long as this is the highest or lowest bid and the identity of the corresponding bidder is also observed. The following result follows directly from prior results (Berman (1963)) for the closely related competing risks model with asymmetric independent risks.

Proposition 2 *In the asymmetric IPV model, assume that each $F_{U_i}(\cdot)$ is continuous. Then each $F_{U_i}(\cdot)$ is identified if either of the following hold:*
(a) In a second-price auction, the highest bid ($B^{(n:n)}$) and the identity of the winner are observed.
(b) In either type of auction, the lowest bid and identity of the lowest bidder are observed.

Proof. See Theorem 7.3.1 and Remark 7.3.1 in Prakasa-Rao (1982). □

When there are only two bidders, the lowest bid is equal to the transaction price, so part (b) provides identification from the transaction price and the identity of the loser. With more than two bidders, the transaction price is an intermediate order statistic, and if any bids are unobserved in an auction data set, the lowest bids are most likely to be missing. Unfortunately, the approaches used to establish identification based on extremal order statistics do not extend immediately to non-extremal order statistics such as the transaction price. However, with one additional observed bid and two more observed bidder identities in each auction, identification follows.

Proposition 3 *In the asymmetric IPV model, assume that each $F_{U_i}(\cdot)$ has a positive density on the interior of its support and that $\forall i,j$, $\{suppF_{U_i}(\cdot) \cap suppF_{U_j}(\cdot)\} \neq \emptyset$. Then each $F_{U_i}(\cdot)$ is identified if $n \geq 3$, the identities of the top three bidders are observed, and the transaction price and next-highest bid (i.e., $B^{(n-1:n)}$ and $B^{(n-2:n)}$) are observed.*

[16] Haile and Tamer (2000) propose an alternative approach to inference and testing motivated by this observation.

Proof. By assumption we observe

$$\Pr(I^{(n:n)} = i,\ I^{(n-1:n)} = j,\ U^{(n-1:n)} \le x) = \int_{-\infty}^{x} \int_{u_j}^{\infty} \prod_{l \ne i,j} F_{U_l}(u_l)\, dF_{U_i}(u_i) dF_{U_j}(u_j).$$

So (where the derivative exists)

$$\frac{\partial}{\partial x} \Pr(I^{(n:n)} = i,\ I^{(n-1:n)} = j,\ U^{(n-1:n)} \le x) = (1 - F_{U_i}(x)) f_{U_j}(x) \prod_{l \ne i,j} F_{U_l}(x).$$

is also observed. Similarly, for $y \ge x$, we observe

$$\frac{\partial^2}{\partial x \partial y} \Pr \left(\begin{array}{c} I^{(n:n)} = i,\ I^{(n-1:n)} = j,\ I^{(n-2:n)} = k, \\ U^{(n-1:n)} \le y,\ U^{(n-2:n)} \le x \end{array} \right) = (1 - F_{U_i}(y)) f_{U_i}(y) f_{U_k}(x) \prod_{l \ne i,j,k} F_{U_l}(x).$$

The ratio of these two expressions is

$$\frac{(1 - F_{U_i}(y)) f_{U_j}(y) f_{U_k}(x)}{(1 - F_{U_i}(x)) f_{U_j}(x) F_{U_k}(x)}.$$

Taking the limit as x approaches y from below yields $\frac{f_{u_k}(x)}{F_{u_k}(x)}$, which uniquely identifies $F_{u_k}(\cdot)$. $\square$

3.3 Conditionally Independent Private Values

If the IPV model is rejected, it is natural to consider the alternative that the private values are correlated. We begin with a model where the correlation among private values takes the special form of the CIPV model or the more restrictive CIPV-I model.[17] It is natural to ask whether the CIPV and CIPV-I models impose significant restrictions relative to the (affiliated and exchangeable) private values model. This question is of independent interest since these are common assumptions in the auctions literature (see, for example, Li, Perrigne, and Vuong (1999)). The de Finetti theorem states that any infinite exchangeable sequence can be represented as a mixture of i.i.d. sequences.[18] Hence, if $U_1, \ldots, U_n$ are exchangeable for all n, the CIPV assumption can be made without loss of generality. However, because we consider a finite number of bidders, conditional independence is a restriction on the private values model. Even if $\mathbf{U}$ is affiliated and exchangeable, conditional independence does not necessarily hold (Shaked (1979)). However, as an approximation of arbitrary affiliated distributions, even the more restrictive CIPV-I model may fit reasonably well: it can always match the mean and variance of U_i, as well as the covariances among (U_i, U_j), and it allows partial flexibility for higher moments.

[17] To understand how CIPV-I differs from CIPV, notice that CIPV-I rules out the possibility that the distribution of A_i depends on V.

[18] More precisely, if $\{X_n\}_{n=1}^{\infty}$ are exchangeable random variables, there exists a σ-algebra $\mathcal{G}$ of events such that for all $m \ge 1$, $\Pr(X_1 < x_1, \ldots, X_n < x_m) = \int \prod_{j=1}^{m} \Pr(X_j < x_j | \mathcal{G})\, d\Pr(\mathcal{G})$. See, e.g., Chow and Teicher (1997).

Proposition 4 *The CIPV-I model (normalized so that $E[A_i] = 0$) can fit the first two moments of any exchangeable and affiliated distribution $F_{\mathbf{U}}(\cdot)$. However, it places a testable restriction on the first through third moments.*

Proof. First, observe that the CIPV-I structure induces affiliated utilities. For a given distribution $F_{\mathbf{U}}(\cdot)$, suppose that $E[U_i]$, $\mathrm{var}(U_i)$, and $\mathrm{cov}(U_i, U_j)$ are given. By exchangeability, these values are the same for all choices of $i \neq j$, and by affiliation, $\mathrm{cov}(U_i, U_j) \geq 0$. Then we can find distributions $F_{\mathbf{A}}(\cdot)$ and $F_V(\cdot)$ such that: $E[V] = E[U_i]$, $\mathrm{var}(V)$=$\mathrm{cov}(U_i, U_j)$, and $\mathrm{var}(A_i)$= $\mathrm{var}(U_i)-\mathrm{cov}(U_i, U_j)$ (implying that $\mathrm{var}(U_i)$ =$\mathrm{var}(A_i)$+$\mathrm{var}(V)$, as required). This is possible because $\mathrm{var}(U_i) \geq \mathrm{cov}(U_i, U_j) \geq 0$. For the third moments, the CIPV-I model implies that

$$E[U_i^2 U_j] - E[U_i U_j U_k] = E[V]E[A_i^2] = E[U_i] \cdot (var(U_i) - cov(U_i, U_j))\,.$$

This restriction is testable. □

3.3.1 Observing all Bids in a Second-Price Auction

This section shows that identification of the CIPV-I model is possible (up to a location normalization) if all bids from a second-price auction are observed. Li, Perrigne, and Vuong (1999) analyze identification of the CIPV-I model in first-price sealed-bid auctions. In the CIPV-I model, each A_i is analogous to an i.i.d. measurement error on the variable V, enabling direct application of identification results from the literature on measurement error (Kotlarski (1966), Prakasa-Rao (1992), Li and Vuong (1998)). In second-price and ascending auctions, observed bids correspond to realizations of order statistics $U^{(j:n)} = A^{(j:n)} + V$, with j from a subset of $\{1, \ldots, n\}$. Therefore the corresponding "measurement errors" $\{A^{(j:n)}\}$ are not independent. This precludes application of prior results unless all order statistics are observed (which is impossible in an ascending auction).[19] Note, however, that when the asymmetric model is identified, it is possible to test the restriction that the A_i are identically distributed across bidders.

Proposition 5 *In the CIPV-I model, assume that $\forall i$ the characteristic functions $\psi_V(\cdot)$ and $\psi_{A_i}(\cdot)$ of the random variables V and A_i are nonvanishing. If all bids are observed in a second-price auction, then:*

(i) The symmetric CIPV-I model is identified.

(ii) The asymmetric CIPV-I model is identified if, in addition, all bidder identities are observed.

[19] Although it seems plausible that since each $A^{(j:n)}$ is an order statistic of an i.i.d. sample from a common parent distribution, there may be sufficient structure to identify the model from only two order statistics $U^{(j:n)}$, $U^{(k:n)}$, we do not present such a result here. It is interesting to note that the difference $U^{(j:n)} - U^{(k:n)} = A^{(j:n)} - A^{(k:n)}$ does not identify $F_{\mathbf{A}}(\cdot)$ up to location (Arnold et al. (1992, p. 143)).

Proof. Follows as in Kotlarski (1966), or as applied to first-price auctions by Li, Perrigne, and Vuong (2000). □

3.3.2 Ex Post Observation of V

An alternative approach to identification and testing in the CIPV and CIPV-I models is available if the ex post realization of V is observed. In a second-price or ascending auction with private values, equilibrium bidding behavior does not depend on whether or not V is ex ante common knowledge among the bidders. Thus, V might be some publicly observed auction-specific covariate, such as the appraised value of an item up for auction or the (common knowledge) cost of raw materials for a construction contract. When V is observed, the CIPV model (and thus, the nested CIPV-I model) is identified from observation of the transaction price. It is testable if more than one bid is observed in each auction.

Proposition 6 *In the CIPV model, suppose that the realization of V is observed.*
(i) If one bid is observed in each auction, $F_V(\cdot)$ and $F_{\mathbf{A}}(\cdot|v)$ are identified for all v.
(ii) If at least two bids are observed in auction, the CIPV model is testable.

Proof. (i) Given v, $a^{(j:n)} = u^{(j:n)} - v = b^{(j:n)} - v$. Identification of $F_{\mathbf{A}}(\cdot|v)$ for each observed v then follows from Proposition 1. (ii) Under the assumptions of part (i), a consistent estimate of $F_{\mathbf{A}}(\cdot|v)$ can be constructed using (3) and standard nonparametric smoothing techniques. With observation of two bids in each auction, two estimates of $F_{\mathbf{A}}(\cdot|v)$ can be constructed and their asymptotic equality tested. □

3.3.3 Auction-Specific Covariates

In many private-value auctions, theory points to more than one source of correlation in bidder values; for example, in timber auctions, the species of timber and tract size are typically observed. However, in some applications it may be the case that conditional on such variables, the remaining variation in private values is independent across bidders. Formally, we can generalize the CIPV and CIPV-I models to allow for a vector of auction-specific variables, W_0, that shift bidder values and are observed to the econometrician.[20] In a generalization of the CIPV model, we assume $\mathbf{A}$ is independent conditional on W_0, and analyze identification of $F_{\mathbf{A}}(\cdot|w_0)$, the joint distribution of $\mathbf{A}$ given $W_0 = w_0$. A more restrictive model, analogous to CIPV-I, assumes that $(\mathbf{A}, W_0)$ is independent, and that W_0 affects bidder utilities through an unknown function $g_0(\cdot)$.

[20] Note that since this structure permits $V \in W_0$, we can assume $V = 0$ without loss of generality.

Proposition 7 *Normalize $V \equiv 0$, and suppose that W_0 and the transaction price are observed.*
(i) If $\mathbf{A}$ is independent conditional on W_0, then $F_{\mathbf{A}}(\cdot|w_0)$ is identified for all w_0.
(ii) If $(\mathbf{A}, W_0)$ are independent, $F_{\mathbf{A}}(\cdot|w_0)$ has density $f_{\mathbf{A}}(\cdot|w_0)\,\forall w_0$, and $U_i = g_0(W_0) + A_i$, where the unknown function $g_0{:}supp[W_0] \to \mathbb{R}$ is differentiable, then for all w_0, $F_{\mathbf{A}}(\cdot|w_0)$ and $g_0(\cdot)$ are identified up to a location normalization.
(iii) If at least two bids are observed in each auction, the models described in (i) and (ii) are testable.

Proof. (i) Apply Proposition 1 for each w_0. (ii) Note that $\Pr(B^{(j:n)} \leq b|W_0) = \Pr(A^{(j:n)} \leq b - g_0(W_0))$. Assume for simplicity that W_0 is a scalar (the argument is the same when this is not the case). Then (where the derivatives exist)

$$\frac{\frac{\partial}{\partial w_0}\Pr(A^{(j:n)} \leq b - g_0(w_0))}{\frac{\partial}{\partial b}\Pr(A^{(j:n)} \leq b - g_0(w_0))} = -g_0'(w_0).$$

This identifies $g_0(\cdot)$ up to a location normalization. Then, for each w_0, $\Pr(A_i \leq b - g_0(w_0))$ is determined through the relationship (3), which identifies $F_A(\cdot|w_0)$. (iii) Following the same approach, the additional bids can be used to identify $F_A(\cdot|w_0)$, and equality can be tested. □

3.4 Unrestricted Private Values

Our approaches to identification and testing of the CIPV and CIPV-I models require potentially strong independence and/or observability assumptions. Moreover, even if the CIPV-I model is rejected, Proposition 4 implies that we cannot rule out the private values model altogether, even if we assume affiliation and exchangeability. It is straightforward to see that for fixed n, any set of observed bids can be rationalized in the private values framework (Laffont and Vuong (1996)): simply let the distribution of values be equal to the distribution of bids. Thus, the PV model is identified from observation of all bids in a sealed-bid auction, but untestable without further information. This raises two questions. First, is the unrestricted PV model identified when some bids are unobserved? And second, what additional information will permit identification and/or testing of the PV model? We address each of these questions below.

3.4.1 The PV Model Is Not Identified from Incomplete Sets of Bids

We begin by showing that, without further information, the unrestricted PV model is not identified in an ascending auction, and it is not identified in a second-price auction unless all bids are observed. This is true even under the exchangeability assumption.. This result is a generalization of a classic result from the literature on competing risks. Cox (1959) and Tsiatis (1975) show that a joint distribution of competing risks is not identified from the lowest order statistic. We show that this

result can be extended to cases in which any combination of order statistics is observed, even when the value distribution is restricted to be exchangeable, so long as not all are observed. Since the method of proof of the prior literature cannot be applied directly, we proceed by constructing a counter-example.[21]

Proposition 8 *Consider the PV model and assume $F_{\mathbf{U}}(\cdot)$ is exchangeable. $F_{\mathbf{U}}(\cdot)$ is not identified from the vector of bids in an ascending auction; further, it is not identified in a second-price auction unless all bids are observed.*

Proof. Suppose that $[0,4]^n$ is in the interior of the support of $\mathbf{X}$ and that $f_{\mathbf{X}}(\cdot)$ is positive throughout this region. Suppose that for some $k \in \{1,\ldots,n\}$ a subset of $\{U^{(j:n)} : j \neq k\}$ is observed but $U^{(k:n)}$ is unobserved. Define a set of partitions of bidder indices

$$\mathcal{S}^k = \{(S_1, S_{k-1}, S_{n-k}) : S_1 \cup S_{k-1} \cup S_{n-k} = \{1,\ldots,n\}, |S_1| = 1, |S_{k-1}| = k-1, |S_{n-k}| = n-k\}.$$

Then, for $S \in \mathcal{S}^k$ and $0 < \varepsilon < 1/2$, define $c(\mathbf{u}; S, \varepsilon) \equiv$

$$\begin{array}{l}\mathbf{1}\{u_i \in [3-\varepsilon, 3+\varepsilon], i \in S_1\} \cdot \mathbf{1}\{u_i \in [1-\varepsilon, 1+\varepsilon], \forall i \in S_{k-1}\} \cdot \mathbf{1}\{u_i \in [4-\varepsilon, 4+\varepsilon], \forall i \in S_{n-k}\} \\ -\mathbf{1}\{u_i \in [2-\varepsilon, 2+\varepsilon], i \in S_1\} \cdot \mathbf{1}\{u_i \in [1-\varepsilon, 1+\varepsilon], \forall i \in S_{k-1}\} \cdot \mathbf{1}\{u_i \in [4-\varepsilon, 4+\varepsilon], \forall i \in S_{n-k}\}\end{array}$$

For sufficiently small $\gamma > 0$, $\breve{f}_{\mathbf{U}}(\cdot) \equiv f_{\mathbf{U}}(\cdot) + \gamma \sum_{S \in \mathcal{S}^k} c(\cdot; S, \varepsilon)$ is a probability density, with the function c shifting probability weight from some regions to others. For example, if $k = n-1$, probability weight shifts from the neighborhood of $(1,\ldots,1,2,4)$ to the neighborhood of $(1,\ldots,1,3,4)$, and similarly for all permutations of the vector pairs.

Observe that $\frac{\partial}{\partial x}\Pr(U^{(m:n)} \leq x) =$

$$n\binom{n}{n-m}\int_{\mathbf{u}_{-m}}\left(\begin{array}{c}\mathbf{1}\{u_i \leq x, i = 1,\ldots,m-1\} \cdot \mathbf{1}\{u_i \geq x, i = m+1,\ldots,n\} \\ \cdot f_{\mathbf{U}}(u_1,\ldots,u_{m-1},x,u_{m+1},\ldots,u_n)\end{array}\right) d\mathbf{u}_{-m}.$$

But, for $\varepsilon < 1/2$, all x, all $S \in \mathcal{S}^k$, and all $m \neq k$,

$$\int_{\mathbf{u}_{-m}} \mathbf{1}\{u_i \leq x, i = 1,\ldots,m-1\}\mathbf{1}\{u_i \geq x, i = m+1,\ldots,n\}c((\ldots,u_{m-1},x,u_{m+1},\ldots); S, \varepsilon)d\mathbf{u}_{-m} = 0.$$

Thus, $\frac{\partial}{\partial x}\Pr(U^{(m:n)} \leq x)$ is unchanged when $f_{\mathbf{U}}(\cdot)$ is replaced with $\breve{f}_{\mathbf{U}}(\cdot)$, and the probability distribution based on the density $\breve{f}_{\mathbf{U}}(\cdot)$ is observationally equivalent to $F_{\mathbf{U}}(\cdot)$. □

[21] Note that this result does not impose affiliation of $\mathbf{U}$. Affiliation is a delicate condition, and if it holds weakly, it is potentially disturbed by small perturbations of the distribution. The result can be generalized to the case where we restrict $\mathbf{U}$ to be affiliated but not independent, as long as we take a smooth, "small enough" perturbation when constructing the counter-example.

3.4.2 Tests of the Symmetric and Asymmetric PV Models

We have shown that the unrestricted PV is not identified in general. Despite this, we now show that the model has testable implications, even in situations where the underlying value distribution is not identified. Our approach requires exogenous variation in the number of bidders, as well as the observation of multiple adjacent bids—e.g., the transaction price $B^{(n-1:n)}$ and the second-highest losing bid $B^{(n-2:n)}$ in an ascending auction, or the highest and second-highest bids in the second-price auction. The following lemma is central to our argument.

Lemma 1 *Consider the PV model. The distribution of $U^{(m-1:m)}$ is identified from the distribution of the order statistics $U^{(j:n)}$, $j = m-1$ to $n-1$, with $m < n$.*

Proof. When the distribution $F_{\mathbf{U}}(\cdot)$ is exchangeable, the following relationship between order statistics can be derived (e.g., David (1981, p. 105)):

$$\frac{n-r}{n} F_U^{(r:n)}(u) + \frac{r}{n} F_U^{(r+1:n)}(u) = F_U^{(r:n-1)}(u). \tag{4}$$

So, from the distributions of $U^{(j:n)}$, $j = m-1$ to $n-1$, we can compute the distributions of $U^{(m-1:n-1)}$ to $U^{(n-2:n-1)}$ using the above formula. If $m = n-1$, this completes the argument. Otherwise, repeating the exercise gives the result. □

The intuition behind (4) is straightforward. When dropping one bidder exogenously (i.e., at random) from a set of n bidders, in each auction, there is a $\frac{r}{n}$ chance that the dropped bidder is one of the bottom r bidders, and a $\frac{n-r}{n}$ chance that the bidder is one of the top $n-r$ bidders. Using these probabilities as weights, the distribution of $U^{(r:n-1)}$ is a weighted average of the distribution of $U^{(r+1:n)}$ and $U^{(r:n)}$.

Proposition 9 *The PV model is testable if we observe the transaction price $B^{(m-1:m)}$ from auctions with $m \geq 2$ bidders and bids $B^{(m-1:n)}, \ldots, B^{(n-1:n)}$ from auctions with $n > m$ bidders. In the case of a second-price auction, it is also sufficient to observe $B^{(m:m)}$ at auctions with m bidders and bids $B^{(m:n)}, \ldots, B^{(n:n)}$ from the n-bidder auctions.*

Proof. Given the equilibrium bidding strategies, observing the bids $B^{(m-1:n)}, \ldots, B^{(n-1:n)}$ and $B^{(m-1:m)}$ is equivalent to observing the order statistics $U^{(m-1:n)}, \ldots, U^{(n-1:n)}$ and $U^{(m-1:m)}$. We can compute the distribution of $U^{(m-1:m)}$ using only the order statistics $U^{(m-1:n)}$ to $U^{(n-1:n)}$ following Lemma 1. Under the private values assumption, this distribution must be the same as that observed directly. The same argument applies to the case in which the order statistics $U^{(m:n)}$ to $U^{(n:n)}$ and $U^{(m:m)}$ are observed. □

This result implies that the PV model is testable whenever we observe the top two (or the second and third highest) bids from auctions with n and $n-1$ bidders, holding all else fixed. Note that Proposition 9 relies critically on symmetry of the bidders, i.e., on the assumption that the distribution of bidder values is exchangeable. However, we can extend the result to the case in which distributions of private values are completely unrestricted, as long as we observe the identities of the participating bidders. The following result exploits the arguments in Balasubramanian and Balakrishnan (1994), who note that by taking random draws from samples of arbitrary random variables, one can obtain a set of exchangeable random variables whose distributions must obey the recurrence relation (4).

Proposition 10 *Take any $\mathcal{P}_n \subset \mathbb{N}$ with $|\mathcal{P}_n| = n \geq 3$ such that the probability that $\mathcal{P}_n$ is the set of participating bidders is positive. If for some $m < n$, $m \geq 2$, there is positive probability of participation by every $\mathcal{P}_m \subset \mathcal{P}_n$ such that $|\mathcal{P}_m| = m$, then if we observe bids $B^{(m-1:n)}, \ldots, B^{(n-1:n)}$ and bidder identities in auctions with n bidders and the transaction price and bidder identities in auctions with m bidders, the (unrestricted, asymmetric) private values model is testable.*[22]

Proof. Let $U_1, \ldots, U_n$ be the random variables corresponding to the valuations of the bidders in $\mathcal{P}_n$, and let $Y_1, \ldots, Y_m$ be a sample of size m drawn without replacement from $\{U_1, \ldots, U_n\}$ using a discrete uniform distribution. Then $Y_1, \ldots, Y_m$ are exchangeable. Define

$$\overline{F}_U^{(r:m)}(u) = \frac{1}{\binom{n}{m}} \sum_{\mathcal{P}_m \subset \mathcal{P}_n : |\mathcal{P}_m| = m} F_U^{(r:\mathcal{P}_m)}(u)$$

where $F_U^{(r:\mathcal{P}_m)}(\cdot)$ is the distribution of the rth order statistic from $\{U_i, i \in \mathcal{P}_m\}$. Exchangeability implies that the distributions of the order statistics of $\{Y_1, \ldots, Y_m\}$ must satisfy (4). Since the distribution of the rth order statistic of $\{Y_1, \ldots, Y_m\}$ equals $\overline{F}_U^{(r:m)}(\cdot)$, for $r < n$ this gives

$$\frac{n-r}{n} \overline{F}_U^{(r:n)}(y) + \frac{r}{n} \overline{F}_U^{(r+1:n)}(y) = \overline{F}_U^{(r:n-1)}(y). \tag{5}$$

Since $\overline{F}_U^{(r+1:n)}(u) = F_U^{(r+1:n)}(u)$, this simplifies to

$$\frac{n-r}{n} F_U^{(r:n)}(u) + \frac{r}{n} F_U^{(r+1:n)}(u) = \overline{F}_U^{(r:n-1)}(u). \tag{6}$$

If $m = n-1$, (6) can be tested directly. For $m < n-1$, repeated application of (6) enables testing of (5). □

[22] As with Proposition 9, in the case of a second-price auction, it is also sufficient to observe all bidder identities and $B^{(m:m)}$ at auctions with m bidders and bids $B^{(m:n)}, \ldots, B^{(n:n)}$ from the n-bidder auctions.

3.4.3 Identification and Testing Using Bidder-Specific Covariates

The last section provided results for testing but not identification. When bidder-specific covariates are available, identification may also be possible.[23] To show this, we adapt approaches from the literatures on competing risks (e.g., Heckman and Honoré (1989), Han and Hausman (1990)) and the Roy model (Heckman and Honoré (1990)). While Heckman and Honoré (1990) point out that the Roy model and the competing risk model share a similar structure, the relationship of these model to auctions has not been previously exploited.

These prior literatures are tailored to cases in which either the lowest or highest order statistic is observed. Observing extreme order statistics (minima or maxima) creates a relatively simple inference problem since the distribution of an extreme order statistic provides direct information about the joint distribution of values. For example, if $U_i = g_i(W_i) + A_i$, then

$$\Pr(B^{(n:n)} \leq b|\mathrm{w}) = F_{\mathbf{A}}(b - g_1(w_1), \ldots, b - g_n(w_n)).$$

Inference from non-extremal order statistics is more difficult; however, when we observe either the highest bid in a second-price auction, or the lowest bid in an ascending auction, existing results can be applied with only minor modification. Analogous to the prior literature, we also need to observe the identity of the auction winner (loser).[24]

Proposition 11 *Consider the asymmetric PV model, normalizing $V \equiv 0$. Assume (a) $U_i = g_i(W_i) + A_i \forall i$; (b) $\mathbf{A}$ is continuously distributed with support $\mathbb{R}^n$; (c) (A_i, W_j) are independent for all i, j, and (d) the support of $(g_1(W_1), \ldots, g_n(W_n))$ is $\mathbb{R}^n$. Then each $g_i(\cdot)$, $i = 1, \ldots, n$, is identified, and $F_{\mathbf{A}}(\cdot)$ is identified up to a location normalization if either (i) at a second-price auction, the winner's identity and bid are observed; or (ii) at either type of auction, the lowest bidder's identity and bid are observed.*

Proof. The proof for the two-bidder case is identical to that for the two-sector Roy model given in Theorem 12 of Heckman and Honoré (1990). The proof when $n > 2$ is identical to that for a multi-sector Roy model. This requires a straightforward extension, which we omit. □

Proposition 11 does not require exchangeability. It does, however, impose fairly stringent conditions on the variation in the bidder covariates $\mathbf{W}$. Our next result relies on similarly strong

[23] Examples of such covariates include the distance from the firm to a construction site or tract of timber, a firm's backlog of contracts won in previous auctions, or measures of demand in the home market of bidders at wholesale used car auctions.

[24] An alternative identification result can be given if bidder identities are unobserved but there is sufficient variation in $g(W_i)$ to effectively "pick" the winner (loser) through the choice of w. More precisely, if the winner's bid is observed, and $\lim_{w_i \to -\infty} g_i(w_i) = -\infty$ at a sealed bid auction, or the lowest bid is observed and $\lim_{w_i \to \infty} g(w_i) = \infty$, then each $g_i(\cdot)$ $i = 1, \ldots, n$ is identified, and $F_{\mathbf{A}}(\cdot)$ is identified up to a location normalization

conditions on **W**, but shows that even if only the transaction price (the second-highest value) is observed, it is possible to uncover the underlying joint distribution of values.

Proposition 12 *Consider the asymmetric PV model, normalizing $V \equiv 0$. Suppose hypotheses (a)–(d) of Proposition 11 hold and that $F_{\mathbf{A}}(\cdot)$ has a differentiable density. Further assume that for each $i = 1, \ldots, n$, $g_i(\cdot)$ is differentiable, $g_i'(\cdot)$ is absolutely continuous, and $\lim_{w_i \to -\infty} g_i(w_i) = -\infty$. Then $F_{\mathbf{A}}(\cdot)$ and each $g_i(\cdot)$, $i = 1, \ldots, n$, are identified from the transaction price.*

Proof. Let $z_i = b - g_i(w_i)$. First, observe that (abusing notation for zero-probability events to simplify exposition)

$$\frac{\partial}{\partial b}\Pr(B^{(n-1:n)} \le b\,|\mathbf{w}) = \sum_{i=1}^{n}\sum_{j \ne i} \Pr\left(A_i \ge z_i,\ A_j = z_j,\ A_k \le z_k\ \forall k \ne i, j\right)$$

$$= \sum_{i=1}^{n}\sum_{j \ne i} \left[\Pr\left(A_j = z_j,\ A_k \le z_k\ \forall k \ne i, j\ \right) - \Pr\left(A_j = z_j,\ A_k \le z_k\ \forall k \ne j\ \right)\right]$$

Then

$$\frac{\partial^n}{\partial w_1 \cdots \partial w_n}\left(\frac{\partial}{\partial b}\Pr(B^{(n-1:n)} \le b\,|\mathbf{w})\right) = -(n-1)\prod_{j=1}^{n}(-g_j'(w_j))\sum_{j=1}^{n}\frac{\partial}{\partial a_j} f_{\mathbf{A}}(\mathrm{a})\Big|_{\mathbf{a}=\mathbf{z}}.$$

Observe that

$$\frac{\partial}{\partial b}\left(\frac{\partial^n}{\partial w_1 \cdots \partial w_n} F_{\mathbf{A}}(b - g_1(w_1), \ldots, b - g_n(w_n))\right) = \prod_{j=1}^{n}(-g_j'(w_j))\sum_{j=1}^{n}\frac{\partial}{\partial a_j} f_{\mathbf{A}}(\mathrm{a})\Big|_{\mathbf{a}=\mathbf{z}}.$$

Thus (using $\lim_{b \to -\infty} F_A^{(n:n)}(b) = F_A^{(n-1:n)}(b) = 0$ and $\lim_{w \to (-\infty, \ldots, -\infty)} F_{\mathbf{A}}(b - g_1(w_1), \ldots, b - g_n(w_n)) = 1$), $F_{\mathbf{A}}(b - g_1(w_1), \ldots, b - g_n(w_n))$ is identified from the observable $\Pr(B^{(n-1:n)} \le b|\mathbf{W})$ by the relation

$$F_{\mathbf{A}}(b - g_1(w_1), \ldots, b - g_n(w_n)) =$$
$$1 - \frac{1}{n-1}\int \cdots \int_{\tilde{\mathbf{w}}=(-\infty,\ldots,-\infty)}^{\mathbf{w}} \left(\frac{\partial^n}{\partial \tilde{w}_1 \cdots \partial \tilde{w}_n}\Pr(B^{(n-1:n)} \le b\,|\tilde{\mathbf{w}})\right) d\tilde{\mathbf{w}}.$$

Given that the composite function $F_{\mathbf{A}}(b - g_1(w_1), \ldots, b - g_n(w_n))$ is identified, we now consider the separate identification of $F_{\mathbf{A}}(\cdot)$ and $g_1(\cdot), \ldots, g_n(\cdot)$. Note that $\lim_{w_{-i} \to (-\infty, \ldots, -\infty)} F_{\mathbf{A}}(b - g_1(w_1), \ldots, b - g_n(w_n)) = F_{A_i}(b - g_i(w_i))$. Varying b and w_i then identifies $g_i(\cdot)$. Repeat the argument for each i. Then, with knowledge of the $g_i(\cdot)$'s, we can identify $F_{\mathbf{A}}(\cdot)$ at any point $(a_1, \ldots, a_n)$ through appropriate choices of b and $\mathbf{w}$. □

The main part of the proof shows how the distribution of the transaction price and variation in the $\mathbf{w}$ identify $F_{\mathbf{A}}(b - g_1(w_1), \ldots, b - g_n(w_n))$. The second part of the proof show how to identify

the functions $g_1(\cdot), \ldots, g_n(\cdot)$. The strategy in this second step entails taking extreme values of $\mathbf{w}_{-i}$, and then varying b and w_i. From a practical standpoint, this strategy has disadvantages, as it relies heavily on observations in the tails of the distribution of $\mathbf{W}$. However, an alternative strategy is available to identify the functions $g_i(\cdot)$ if the identity of the highest and second-highest bidders are observed in addition to the transaction price. We briefly sketch this approach.[25]

Let $I^{(j:n)}$ denote the identity of the bidder making bid $B^{(j:n)}$. From the identity of the first and second-highest bidders, for each i,j, $\Pr(I^{(n:n)} = i,\ I^{(n-1:n)} = j|\mathbf{w})$ is identified and equal to

$$\Pr\left(A_i + g_i(w_i) \geq\ A_j + g_j(w_j) \geq\ A_k + g_k(w_k)\ \forall k \neq i, j\right).$$

We can then define the set

$$S_{\bar{w}} = \left\{\mathbf{w} : \forall i, j, \Pr(I^{(n:n)} = i, I^{(n-1:n)} = j|\mathbf{w}) = \Pr(I^{(n:n)} = i, I^{(n-1:n)} = j|\bar{\mathbf{w}})\right\}.$$

Next, observe that the following is identified for each (i, j):

$$\begin{aligned}
&\Pr\left(B^{(n-1:n)} \leq b, I^{(n:n)} = i, I^{(n-1:n)} = j\,|\mathbf{w}\right) = \Pr\left(A_j + g_j(w_j) \leq b,\ I^{(n:n)} = i, I^{(n-1:n)} = j\,|\mathbf{w}\right) \\
&= \Pr\left(A_j + g_j(w_j) \leq b\ \left|I^{(n:n)} = i, I^{(n-1:n)} = j, \mathbf{w}\right.\right) \cdot \Pr\left(I^{(n:n)} = i, I^{(n-1:n)} = j\,|\mathbf{w}\right)
\end{aligned}$$

For $\mathbf{w} \in S_{\bar{w}}$, $\Pr\left(I^{(n:n)} = i, I^{(n-1:n)} = j\,|\mathbf{w}\right)$ is constant. So for any b and any $\mathbf{w}' \neq \mathbf{w}$ in $S_{\bar{w}}$, the b' that solves

$$\Pr(B^{(n-1:n)} \leq b, I^{(n:n)} = i, I^{(n-1:n)} = j\,|\mathbf{w}) = \Pr(B^{(n-1:n)} \leq b', I^{(n:n)} = i, I^{(n-1:n)} = j\,|\mathbf{w}').$$

must satisfy $g_j(w_j') - g_j(w_j) = b' - b$. As long as $\{w_j \in S_{\bar{w}}\} = \mathbb{R}$, varying $\mathbf{w}$ and $\mathbf{w}'$ identifies $g_j(\cdot)$ up to location.

Assuming that consistent estimators can be constructed using each of the identification strategies in Propositions 11 and 12,[26] a test of the PV model can be implemented by comparing them.

Corollary 1 *Suppose that the hypotheses of Proposition 12 hold. Then the asymmetric PV model is testable if each of the following is observed: (a) the transaction price; (b) either the winner's bid and identity at a second-price auction, or the lowest bidder's bid and identity.*

---

[25] The argument follows the proof of Theorem 12 in Heckman and Honoré (1990), but considers the case of non-extreme order statistics in a model with more than two bidders (analogous to more than 2 *sectors* in the Roy model).

[26] We are unaware of work developing nonparametric estimation approaches for competing risks models or the Roy model based on these identification strategies. Doing so is beyond the scope of the present paper.

4 Ascending and Second-Price Auctions with Common Values

4.1 Testing Based on Multiple Bids and Variable Number of Bidders

In Section 3.4.2, we established that the PV model is testable using exogenous variation in the number of bidders (Proposition 9). Proposition 9 relied on a recurrence relation between distributions of order statistics of bids within the PV framework. In the second-price auction, the recurrence relation is replaced with a stochastic dominance relation under the CV model. For example, the distribution of transaction prices from auctions with $n-1$ bidders stochastically dominates a particular convex combination of bid distributions from n-bidder auctions. A similar result holds for the ascending auction, where a recurrence relation between means of order statistics from samples of exchangeable random variables implies a strict ordering of the same combination of means under the CV hypothesis. Both results arise from the winner's curse, which is more severe in auctions with larger numbers of bidders.

Proposition 13 *The CV model is testable if we observe the transaction price $B^{(m-1:m)}$ in auctions with $m \geq 2$ bidders and bids $B^{(m-1:n)}, \ldots, B^{(n-1:n)}$ in auctions with $n > m$ bidders.*

Proof. Assume $m = n-1$ (the argument is similar for other cases). First consider a second-price auction, where (recalling (1)) each player i bids

$$b_i = E[U_i | X_i = \max_{j \neq i} X_j = x_i] \equiv b(x_i; n).$$

If instead of $b(\cdot; n)$, the bidders used a symmetric monotonic bidding strategy that did not depend on n, (4) would imply

$$\frac{2}{n} \Pr(B^{(n-2:n)} \leq b) + \frac{n-2}{n} \Pr(B^{(n-1:n)} \leq b) = \Pr(B^{n-2:n-1} \leq b) \tag{7}$$

since bids would be fixed monotonic transformations of exchangeable signals, implying that the bids were also exchangeable. However, when the equilibrium strategy $b(\cdot; n)$ is used, taking $i = 1$ without loss of generality and exploiting exchangeability, we have

$$\begin{aligned} b(x_1; n) &= E[U_1 | X_1 = X_2 = x_1, X_j \leq x_1, \ j = 3, \ldots, n] \\ &< E[U_1 | X_1 = X_2 = x_1, X_j \leq x_1, \ j = 3, \ldots, n-1] \\ &= b(x_1; n-1) \end{aligned} \tag{8}$$

with the strict inequality following from the fact that $E[U_1 | X_1, \ldots, X_n]$ strictly increases in each X_i, due to strict affiliation of (U_1, X_i). Hence, $b(x_1; n)$ strictly decreases in n, implying the testable stochastic dominance relation

$$\frac{2}{n} \Pr(B^{(n-2:n)} \leq b) + \frac{n-2}{n} \Pr(B^{(n-1:n)} \leq b) > \Pr(B^{n-2:n-1} \leq b) \tag{9}$$

instead of (7).

For an ascending auction, fix a realization of $(X_1, \ldots, X_{n-1})$, with $X^{(n-2:n-1)} = x$. Without loss of generality, suppose it is bidder 2 who has signal x and bidder 1 who has the higher signal. When bidders have these signals in an $(n-1)$-bidder auction, bidders $3, \ldots, n-1$ will (in equilibrium) drop out and reveal their signals before bidders 1 and 2. Bidder 2 then drops out at price

$$\begin{aligned}
b^{(n-2:n-1)} &= E[U_2 \,|\, X_1 = X_2 = x,\, X_j = x_j,\, j = 3, \ldots, n-1] \\
&= E_{X_n}\left[E\left[U_2 \,|\, X_n,\, X_1 = X_2 = x,\, X_j = x_j,\, j = 3, \ldots, n-1\right]\right] \\
&> \Pr(X_n > x \,|x_{-n})\; E\left[U_2 | X_n = X_1 = X_2 = x, X_j = x_j,\, j = 3, \ldots, n-1\right] \\
&\quad + \Pr(X_n \leq x | x_{-n}) E\left[E\left[U_2 \,|X_1 = X_2 = x,\; X_n,\; X_j = x_j,\, j = 3, \ldots, n-1\right] \middle|\; X_n \leq x\right] \\
&\equiv b^+(x_2; x_3, \ldots, x_{n-1}).
\end{aligned}$$

Here $b^+(x_2; x_3, \ldots, x_{n-1})$ is the expected bid of bidder 2 in an auction where an additional bidder n is also included, given the realizations of $X_1, \ldots, X_{n-1}$. To see this, observe that if $x_n < x$, bidder n will drop out before bidder 2, revealing x_n. If $x_n > x$, bidder 2 will drop out before bidder n, with 2's exit price based on an expectation that conditions on all remaining bidders, *including* n, having signal x. Taking expectations over $X_1, \ldots, X_{n-1}$ gives the testable inequality restriction

$$E\left[B^{(n-2:n-1)}\right] > E\left[b^+(X^{(n-2:n-1)}; X_3, \ldots, X_{n-1})\right] = \frac{2}{n} E[B^{(n-2:n)}] + \frac{n-2}{n} E[B^{(n-1:n)}] \quad (10)$$

since, by exchangeability, $\Pr(X_n > X^{(n-2:n-1)}) = \frac{2}{n}$. $\square$

For the ascending auction, a test of the inequality (10) can be implemented using standard one-sided hypothesis tests. Since (9) implies (10), the same test could be used for a sealed-bid auction. Of course, other tests of (9) are possible, including direct tests of the stochastic dominance relation, following McFadden (1989).

Note that Propositions 9 and 13 imply that the observational equivalence between the CIPV and CV models noted in Li, Perrigne and Vuong (1999) is eliminated when one observes exogenous variation in the number of bidders.[27] In addition, each of these propositions implies that the test proposed in the other has power against an important alternative. In particular, the PV alternative requires that (9) and (10) hold with equality rather than inequalities. Note also that if the data indicate that the inequality in (9) or (10) is reversed, this will suggest a violation of one of our more basic maintained assumptions.

One limitation of this result is that, unlike our models of private values, the assumption that bidders know the number of competitors they face plays an important role. While this assumption

[27] Other approaches for empirically distinguishing these models based on observation of all bids at first-price auctions are given in Hendricks, Pinkse and Porter (1999) and Haile, Hong and Shum (2000).

does not seem severe in an ascending auction, it may be restrictive in a sealed-bid auction. We return to discuss the extent to which this assumption can be relaxed in Section 6.1.

Without some additional structure, we are unaware of an approach to identify the CV model. Thus, in the remainder of this section, we consider identification and testing of the pure CV model.

4.2 Pure Common Values

We begin by highlighting several issues that arise when attempting to identify the pure CV model. First, from an economic perspective, the scaling of the signals $\mathbf{X}$ is arbitrary. That is, bidder behavior and utility depends on the information content of the signal, and that is preserved under monotone transformations. Given this, a particularly salient normalization of signals satisfies

$$E[V|X_i = \max_{j \neq i} X_j = x, n] = x. \tag{11}$$

With this normalization (recalling (1)), the optimal bidding strategy in a second-price auction is $b(x_i) = x_i$. While the same logic applies to the lowest bid in an ascending auction, all higher bidders in an ascending auction condition their bids on the drop-out points of lower bidders. For this reason, identification in ascending auctions is much more complex. Because bidder beliefs change as the auction progresses, no single normalization can induce the simple strategy $b(x) = x$ for all bidders in the ascending auction. This will limit the set of data configurations under with any CV model can be identified in ascending auctions.

If all bids are observed in a second-price auction, the normalization (11) implies that the joint distribution of bids is equal to the distribution of $\mathbf{X}$, and thus the distribution of $\mathbf{X}$ is identified trivially.[28] However, if some bids are unobserved, the distribution of observed bids provides information only about certain order statistics of $\mathbf{X}$. We might hope to apply the identification and testing approaches developed for private values models, where the same problem arises, and indeed, we will pursue this. However, the assumptions required are somewhat more subtle. For example, $\mathbf{X}$ generally will not be independent in the CV formulation. Further, although it may be natural to assume that $X_i = V + E_i$ for some E_i (where $\mathbf{E}$ is either independent of V or independent conditional on V), this additive structure may not survive the rescaling (11). To address this concern, we consider two (nested) special cases of the pure CV model:[29]

[28] Further, with pure common values and observability of ex post values, the model can be tested by comparing estimates of $E[V|B_i = \max_{j \neq i} B_j = b_i, n]$ based directly on the observed bids and V to those inferred from bids using bidders' first-order conditions. This is the analog of the approach followed in Hendricks, Pinske, and Porter (1999) for first-price auctions.

[29] Each of these is a special case of the *mineral rights model* described in Milgrom and Weber (1982), where bidders receive signals of the common value v that are independent conditional on v.

Linear Mineral Rights (LMR): $U_i = V$. In addition, for each n there exist two known constants $(C, D) \in \mathbb{R} \times \mathbb{R}_+$ and random variables $(E_1, \ldots, E_n)$ with joint distribution $F_{\mathbf{E}}(\cdot)$ such that, with the normalization $E[V|X_i = \max_{j \neq i} X_j = x, n] = x$, either (i) $X_i = \exp(C) \cdot (V \cdot E_i)^D \ \forall i$, with (E, V, X) non-negative; or (ii) $X_i = C + D(V + E_i) \ \forall i$. Further, conditional on V, the elements of $\mathbf{E}$ are mutually independent.

LMR with Independent Components (LMR-I): In the LMR model, $(V, \mathbf{E})$ are mutually independent.

The LMR model is analogous to the CIPV model, while the LMR-I model is analogous to CIPV-I.[30] However, the functional form assumptions are potentially more demanding in this case. Li, Perrigne, and Vuong (1999) show that several plausible models satisfy LMR-I. Condition (i) in the LMR definition is satisfied if bidders see signals $(Y_1, \ldots, Y_n)$, where $Y_i = V \cdot E_i$, $(V, \mathbf{E})$ are independent, and there exist known constants $(C, D) \in \mathbb{R} \times \mathbb{R}_+$ such that $\ln(E[V|Y_i = \max_{j \neq i} Y_j = y_i]) = C + D \ln(y_i)$. To see this, let $X_i = \exp(C) \cdot Y_i^D$; since this is a monotone transformation of Y_i, it has the same information content. The following are examples: (a) $f(v) \propto \frac{1}{v^\gamma}$, $\gamma \in \mathbb{R}$, whereby $C = \ln\left(\frac{E[E_i^{\gamma-1}|E_i = \max_{i \neq j} E_j]}{E[E_i^{\gamma}|E_i = \max_{i \neq j} E_j]}\right)$ and $D = 1$; (b) there are two bidders, and $(V, \mathbf{E})$ are log-normal. Similarly, the following are examples of case (ii): (a) the prior $f_V(\cdot)$ is flat; (b) there are two bidders, and $(V, \mathbf{E})$ are normally distributed.

We begin by analyzing identification and testing in the LMR-I model. Our main result shows that our results for the CIPV-I model extend to the LMR-I model; as with Proposition 5, the result applies only to second-price auctions where all bids are observed.

Proposition 14 *In the LMR-I model, assume that for all i the characteristic functions $\psi_V(\cdot)$ and $\psi_{A_i}(\cdot)$ of the random variables V and A_i are nonvanishing. If all bids are observed in a second-price auction, the LMR-I model is identified and testable.*

Proof. Consider case (ii) of the LMR-I model. By (1), each bidder i's bid b_i is equal to $C + D(v + e_i)$, from which we can infer $v + e_i$. Hence, each bid $B^{(j:n)}$ reveals $V + E^{(j:n)}$. The proof of Proposition 5 then applies directly. Case (i) follows in the same manner after taking logarithms. □

Observing information about the ex post value of the object enables us to identify a more general pure common values model. In particular, our results for the CIPV model can be generalized to prove identification of the LMR model. However, in the case of ascending auctions our results are fairly weak, requiring that the lowest bid be observed. This restriction is required because after

[30] Identification of the LMR-I model at first-price auctions has been considerd by Li, Perrigne and Vuong (1999). See also Paarsch (1992a), who considers several parametric specifications of the mineral rights model.

the first bidder drops out, the strategies of remaining bidders depend on $B^{(1:n)}$, implying that no normalization yields bid functions equal to the identity function.

Proposition 15 *Consider the LMR model. Suppose that the ex post realization of V is observed. (i) If the transaction price is observed in a second-price auction, $F_{\mathbf{E}}(\cdot|v)$ and $F_V(\cdot)$ are identified up to the normalization (11). If one additional bid is observed in each auction, the model is testable. (ii) If the lowest bid is observed in either a second-price or ascending auction, $F_{\mathbf{E}}(\cdot|v)$ and $F_V(\cdot)$ are identified up to the normalization (11).*

Proof. Consider case (ii) in the definition of the LMR model. By (1), each bid b_i is equal to $C + D(v + e_i)$, from which we can infer $v + e_i$. Hence, the transaction price in a sealed bid auction reveals $v + e^{(n-1:n)}$ while the lowest bid in either type of auction reveals $v + e^{(1:n)}$. In either case, observation of v reveals one order statistic of $\mathbf{E}$, so identification of $F_{\mathbf{E}}(\cdot)$ and testing then follows directly from the arguments of Proposition 6. □

5 First-Price Auctions when Some Bids are Unobserved

Most of our results for PV auctions can be extended to first-price auctions, although in some cases we require that the second-highest bid be observed in addition to the transaction price. As is well known, the first-price auction and descending (Dutch) auction are strategically equivalent. An important difference for the econometrician, however, is that in the descending auction, the winning bid is the only bid made. Hence our results below for the first-price auction that require only the transaction price apply directly to the descending auction.[31] We will restrict attention to affiliated values in this section because this assumption guarantees monotonicity of the equilibrium bidding strategies; however, the assumption is otherwise unrelated to our identification arguments. However, here we do make the additional assumptions that $F_{\mathbf{A}}(\cdot)$ and $F_{\mathbf{X}}(\cdot)$ are strictly increasing on their respective supports and have associated joint densities $f_{\mathbf{A}}(\cdot)$ and $f_{\mathbf{X}}(\cdot)$.

A bidder in a first-price auction views the bids of his opponents as random variables. When bidder i observes $X_i = x_i$, he solves

$$\max_{b_i} \left(E[U_i | X_i = x_i, \max_{j \neq i} B_j \leq b_i] - b_i \right) \Pr(\max_{j \neq i} B_j \leq b_i | X_i = x_i).$$

Define

$$\zeta(x; n) = E[U_i | X_i = x, \max_{j \neq i} X_j = x].$$

[31] Laffont and Vuong (1993) and Elyakime et al. (1994) address parametric identification and estimation of the descending auction model.

Then i's first-order condition implies that a necessary condition for b_i to be an optimal bid is $b_i + \frac{\Pr(\max_{j \neq i} B_j \leq b_i | X_i = x_i)}{\frac{\partial}{\partial b_i} \Pr(\max_{j \neq i} B_j \leq b_i | X_i = x_i)} = \zeta(x_i; n)$. By standard arguments, in an equilibrium to this model, bidding strategies are strictly increasing.[32] Thus, if the bids $(B_1, \ldots, B_n)$ are generated from such an equilibrium, bidder i's bid B_i has the same information content as X_i, so that

$$b_i + \frac{\Pr(\max_{j \neq i} B_j \leq b_i | B_i = b_i)}{\frac{\partial}{\partial z} \Pr(\max_{j \neq i} B_j \leq z | B_i = b_i)\big|_{z=b_i}} = \zeta(x_i; n). \tag{12}$$

This equation, expressing the latent expectation $\zeta(x_i; n)$ in terms of observable bids, has been widely exploited in the literature; Elyakime et al. (1994) first used it to analyze the IPV model, Li, Perrigne, and Vuong (1998) for the affiliated PV model, and Campo, Perrigne and Vuong (2000) for the asymmetric PV model (see Perrigne and Vuong (1999) for a survey). Hendricks, Pinkse, and Porter (1999) and Li, Perrigne, and Vuong (1999) use this approach to identify and estimate models of pure CV auctions.

When all bids are observed (and bidder identities, where relevant), $\Pr(\max_{j \neq i} B_j \leq b | B_i = z)$ is observed, so $F_{\mathbf{U}}(\cdot)$ is identified from the distribution of bids through (12) (Laffont and Vuong (1993, 1996)). In the symmetric IPV model of first-price auctions, Guerre, Perrigne, and Vuong (1995) established that $F_U(\cdot)$ is also identified from the transaction price alone, since the transaction price identifies the distribution $H_{\mathbf{B}}^{(n:n)}(b) = (H_B(b))^n$. Thus, $\Pr(\max_{j \neq i} B_j \leq b_i | B_i = b_i) = \left(H_{\mathbf{B}}^{(n:n)}(b_i)\right)^{\frac{n-1}{n}}$, so the left-hand side of (12) is identified. However, when bidders are asymmetric, another approach is required.

Proposition 16 *Consider the IPV model of the first-price auction.*
(i) If bidders are symmetric and the transaction price is observed, then $F_{\mathbf{U}}(\cdot)$ is identified and the model is testable.
(ii) If bidders are asymmetric and the transaction price and the identity of the winner are observed, then each $F_{U_i}(\cdot)$ is identified and the model is testable.

Proof. (i) Follows from Guerre, Perrigne, and Vuong (1995), Corollary 2. The left-hand side of (12) must be strictly increasing in equilibrium (Laffont and Vuong (1996)), providing a testable restriction on the estimate of this function.

(ii) We observe the distribution of the transaction price, $\Pr(B^{(n:n)} \leq b)$, as well as the identity of the winning bidder, $I^{(n:n)}$. In equilibrium, each bidder's strategy is a strictly increasing function, and

[32] In asymmetric first-price auction models, existence of equilibrium is not straightforward. Athey (2000) shows that as long as each bidder's best response to nondecreasing opponent strategies is itself nondecreasing, a pure strategy Nash equilibrium exists. A sufficient condition for such monotonicity in a PV auction is affiliation. In CV models, the conditions required for monotone best responses are more stringent when there are more than two asymmetric bidders, but Athey (2000) shows that they are satisfied in the LMR model.

the bids are independent. Let $H_{B_i}(\cdot)$ be the marginal distribution of B_i. The proof of Proposition 2 implies that the distributions $H_{B_i}(\cdot)$ are identified. From these distributions we can compute for each bidder $\Pr(\max_{i \neq j} B_j \leq b)$. Then (12) and the joint distribution of $(B^{(n:n)}, I^{(n:n)})$ identify the joint distribution of $(U^{(n:n)}, I^{(n:n)})$. Applying Proposition 2 yields identification of each $F_{U_i}(\cdot)$. Testing follows as in (i). $\square$

Note that the assumptions in part (ii) contrast with existing results for identification of asymmetric PV models (Laffont and Vuong (1996)), which have relied on observation of all bids. As mentioned above, Dutch auctions are strategically equivalent to first-price auctions, and have the feature that only the transaction price is observable; thus, this result will be useful for Dutch auctions as well as first-price auctions where only the winning bid is recorded. Estimation strategies can be adapted from those in, e.g., Guerre, Perrigne, and Vuong (2000). Note that the tests described in the proof of Proposition 16 may not be very powerful—one can easily construct non-equilibrium bidding rules leading to bids that satisfy the required monotonicity. However, additional tests are possible when the top two bids are observed. For example, with symmetric bidders, the approach taken in Proposition 16 can then be applied, replacing $B^{(n:n)}$ with $B^{(n-1:n)}$, yielding a second estimate of $F_U(\cdot)$. This estimate and that based on the transaction price must be asymptotically equivalent.[33]

Although most first-price auction data sets either contain only the transaction price or else all bids, there may be reasons for a real-world auctioneer (or auction participants) to maintain records of the top two bids. For example, in procurement auctions, the top bidder may be disqualified or default before the contract is awarded, and the second-highest bidder might receive the contract instead. Further, auction participants often refer to the difference between the winning bid and the second-highest bid as "money left on the table," which has intuitive appeal as information that is relevant to strategic bidding behavior. The next result shows that using observations of the top two bids, it is possible to compute the equilibrium "markdown," or gap between a bidder's bid and her value, as in (12). This allows us to test the affiliated PV model. Of course, the same testing approach can be applied when all bids are observed.

Proposition 17 *Consider the affiliated PV model of the first-price auction, and suppose that the highest two bids ($B^{(n:n)}$ and $B^{(n-1:n)}$) are observed.*
(i) If bidders are symmetric, then the joint distribution of $(U^{(n:n)}, U^{(n-1:n)})$ is identified, and the

[33] The case in which the lowest bid and bidder identity are observed is analogous. Note that observing the top two bids ($B^{(n:n)}$ and $B^{(n-1:n)}$) also allows for an alternative estimation strategies, based on the the relation $\frac{1}{n}H_b^{(n-1:n)}(b) + \frac{n-1}{n}H_b^{(n:n)}(b) = H_b^{(n-1:n-1)}(b)$, which follows from exchangeability. Given $H_b^{(n-1:n-1)}(b)$, we can apply (12) to $B^{(n:n)}$ and $U^{(n:n)}$; then, from the observed $H_b^{(n:n)}(\cdot)$ we infer the distribution of $U^{(n:n)}$. Finally, (3) can be applied to infer the distribution of U_i from the distribution of $U^{(n:n)}$.

model is testable.

(ii) If bidders are asymmetric and the identity of the winner ($I^{(n:n)}$) is also observed, then the joint distribution of $(U^{(n:n)}, U^{(n-1:n)}, I^{(n:n)})$ is identified, and the model is testable.

Proof. (i) Let $H_{\mathbf{B}}(b_1, \ldots, b_n)$ be the joint distribution of the bids. Taking $i = 1$ without loss of generality

$$\frac{\Pr(\max_{j\neq 1} B_j \leq b_1 | B_1 = b_1)}{\frac{\partial}{\partial x} \Pr(\max_{j\neq 1} B_j \leq x | B_1 = b_1)\big|_{x=b_1}} = \frac{\frac{\partial}{\partial y} \Pr(\max_{j\neq 1} B_j \leq b_1,\ B_1 \leq y)\Big|_{y=b_1}}{\frac{\partial^2}{\partial x \partial y} \Pr(\max_{j\neq 1} B_j \leq x,\ B_1 \leq y)\Big|_{x=y=b_1}}$$

$$= \frac{\frac{\partial}{\partial y} H_{\mathbf{B}}(y, b_1, \ldots, b_1)\Big|_{y=b_1}}{(n-1)\ \frac{\partial^2}{\partial y \partial z} H_{\mathbf{B}}(y, z, x, \ldots, x)\Big|_{x=y=z=b_1}} = \frac{\frac{\partial}{\partial x} \Pr(B^{(n:n)} \leq x)\big|_{x=b_1}}{\frac{\partial^2}{\partial x \partial y} \Pr(B^{(n-1:n)} \leq x,\ B^{(n:n)} \leq y)\Big|_{x=y=b_1}}.$$

The first equality follows by Bayes' rule, canceling terms from the numerator and the denominator. The second equality follows by the definitions and exchangeability; the third equality follows from differentiation and exchangeability. Since the joint distribution of $(B^{(n-1:n)}, B^{(n:n)})$ is observable by assumption, identification of the distribution of $(U^{(n:n)}, U^{(n-1:n)})$ follows from (12). In equilibrium the left-hand side of (12) must be increasing, which is a testable restriction.

(ii) Extending the logic from (i), now we have

$$\frac{\frac{\partial}{\partial y} \Pr(\max_{j\neq 1} B_j \leq b_1,\ B_1 \leq y)\Big|_{y=b_1}}{\frac{\partial^2}{\partial x \partial y} \Pr(\max_{j\neq 1} B_j \leq x,\ B_1 \leq y)\Big|_{x=y=b_1}} = \frac{\frac{\partial}{\partial y} H_{\mathbf{B}}(y, b_1, \ldots, b_1)\Big|_{y=b_1}}{\sum_{j\neq 1} \frac{\partial^2}{\partial y \partial z_j} H_{\mathbf{B}}(y, z_2, \ldots, z_n)\Big|_{z_2=\cdots=z_n=x}}$$

$$= \frac{\frac{\partial}{\partial x} \Pr(B^{(n:n)} \leq x,\ I^{(n:n)} = 1)\big|_{x=b_1}}{\frac{\partial^2}{\partial x \partial y} \Pr(B^{(n-1:n)} \leq x,\ B^{(n:n)} \leq y,\ I^{(n:n)} = 1)\Big|_{x=y=b_1}}.$$

Identification follows from (12) since the last term above is observable; testing follows as in (i). □

Proposition 17 establishes that we can identify the joint distribution of the top two bidder valuations. Although this is not enough to identify $F_{\mathbf{U}}(\cdot)$, it is sufficient for some important policy simulations, including evaluation of alternative reserve prices. This also allows us to apply many of the results derived in Section 3. The following result follows from Propositions 17, together with Propositions 8, 6, 7, and 11.

Corollary 2 *Consider the affiliated PV model of the first-price auction and supposesuppose that the highest two bids are observed. If bidders are allowed to be asymmetric, assume identity of the winner is observed as well. Then:*

(i) $F_{\mathbf{U}}(\cdot)$ is not identified if any bids below the highest two are unobserved.[34]
(ii) Under the additional restrictions of the CIPV model, if V is observed, then $F_{\mathbf{A}}(\cdot)$ is identified.
(iii) If W_0 is observed, $(\mathbf{A}, W_0)$ are independent, and $U_i = g_0(W_0) + A_i$, where the unknown function $g_0 : supp[W_0] \to \mathbb{R}$ is differentiable, then $g_0(\cdot)$ and $F_{\mathbf{A}}(\cdot)$ are identified up to a location normalization.
(iv) If $(W_1, \ldots, W_n)$ are observed, and $U_i = g_i(W_i) + A_i$, then under the additional restrictions of Proposition 11, then each $g_i(\cdot)$, $i = 1, \ldots, n$, is identified, and $F_{\mathbf{A}}(\cdot)$ is identified up to a location normalization.

Further, Proposition 17 implies that the top two bids in a first-price auction give us enough information to test the PV model against the CV alternative.

Corollary 3 *Consider the affiliated PV model of the first-price auction, and suppose that the top two bids are observed in an auction with $n \geq 3$ bidders, while the highest bid is observed in an auction with $n-1$ bidders.*
(i) If bidders are symmetric, the model can be tested against the CV alternative.
(ii) If bidders are permitted to be asymmetric, the model can be tested if, in addition, we observe the bidder identities corresponding to each of the observed bids.

Proof. (i) By Proposition 17, this information allows us to identify the distributions of $U^{(n-1:n-1)}$, $U^{(n-1:n)}$, and $U^{(n:n)}$ under the PV hypothesis, in which case these distributions must satisfy the recurrence relation (4). Now consider the symmetric CV alternative. By standard arguments, $\zeta(x;m)$ is strictly increasing in x, so the random variables $\zeta(X_i;n)$ are exchangeable. The arguments of Proposition 17 then imply that the distributions of $\zeta(X^{(n:n)};n)$, $\zeta(X^{(n-1:n)};n)$, and $\zeta(X^{(n-1:n-1)};n-1)$ are identified. Then, (3) implies

$$\begin{aligned} \frac{n-1}{n}\Pr\left(\zeta(X^{(n:n)};n) \leq z\right) + \frac{1}{n}\Pr\left(\zeta(X^{(n-1:n)};n) \leq z\right) &= \Pr\left(\zeta(X^{(n-1:n-1)};n) \leq z\right) \\ &> \Pr\left(\zeta(X^{(n-1:n-1)};n-1) \leq z\right) \quad (13) \end{aligned}$$

with the inequality following from (8). This inequality is testable.

[34] To see how this follows from Proposition 8, suppose that the true distribution of utilities is $F_{\mathbf{U}}(\cdot)$, the jth bid is unobserved ($j < n-1$), and $\boldsymbol{\beta}$ is the set of equilibrium bidding functions. These together with $F_{\mathbf{U}}(\cdot)$ imply a distribution of bids $G_{\mathbf{B}}(\cdot)$. Now take another value distribution, $\tilde{F}_{\mathbf{U}}(\cdot)$, such that $\tilde{F}_{\mathbf{U}}(\cdot)$ and $F_{\mathbf{U}}(\cdot)$ induce the same distributions of order statistics except for the jth; this exists by Proposition 8. But then, the bidding functions $\boldsymbol{\beta}$ still form an equilibrium. To see this, observe that when player i computes her best response to opponent bidding strategies $\boldsymbol{\beta}_{-i}$, only the joint distribution of the highest two order statistics of the value distribution affect her optimization problem. Since this joint distribution is identical for $\tilde{F}_{\mathbf{U}}(\cdot)$ and $F_{\mathbf{U}}(\cdot)$, player i's best response bid for each value of u_i is unchanged. Because the bidding functions $\boldsymbol{\beta}$ together with the value distribution $\tilde{F}_{\mathbf{U}}(\cdot)$ imply a distribution over bids, $\tilde{G}_{\mathbf{B}}(\cdot)$, where $\tilde{G}_{\mathbf{B}}(\cdot)$ and $G_{\mathbf{B}}(\cdot)$ induce the same distributions of order statistics except for the jth order statistic. Since the jth bid is unobserved, $\tilde{G}_{\mathbf{B}}(\cdot)$ is observationally equivalent to $G_{\mathbf{B}}(\cdot)$. Finally, recall that Proposition 8 does not impose affiliation; as discussed in footnote 21, the result can be generalized to that case.

(ii) Using a similar argument,

$$\Pr\left(U^{(n-1:n-1)} \leq b,\ I^{(n-1:n-1)} = i\right) = \frac{n-1}{n}\Pr\left(U^{(n:n)} \leq b, I^{(n:n)} = i\right) + \frac{1}{n}\Pr\left(U^{(n-1:n)} \leq b, I^{(n-1:n)} = i\right).$$

The terms in this expression are identified (by Proposition 17), so equality can be tested. □

6 Extensions

6.1 Bidder Uncertainty Over the Number of Opponents

6.1.1 Second-Price Auctions

As pointed out above, our analysis of the CV models assumed that bidders know the number of competitors they face. In an ascending auction, this assumption may be natural. In a sealed-bid auction, bidders might not know the number of competitors when submitting their bids.[35] However, as long as bidders observe an informative signal η of the level of competition, our results for the CV model can be extended. Our identification results for the LMR and LMR-I models extend immediately, as long as (1) η replaces n in the LMR definition, and (2) in the case of a first-price auction, the econometrician can distinguish between auctions with different realizations of η.[36] The following result shows that the testing approach of Proposition 13 can still be applied.

Proposition 18 *In the second-price sealed-bid auction, suppose bidders do not know n but observe a public signal η that is strictly affiliated with n (with η unobserved to the econometrician). Then the CV model is testable if we observe the transaction price $B^{(m-1:m)}$ in auctions with m bidders and bids $B^{(m-1:n)}, \ldots, B^{(n-1:n)}$ in auctions with $n > m$ bidders.*

Proof. Let $\pi(\eta|n)$ denote the conditional distribution of the signal η. Assume $m = n-1$ (the argument is similar for other cases). Given signal η, each player i bids

$$b_i = E_n\left[E[U_i|X_i = \max_{j\neq i} X_j = x_i]\Big|\eta\right] \equiv \hat{b}(x_i;\eta).$$

[35] Matthews (1987) and McAfee and McMillan (1987b) provide theoretical analyses of auctions with a stochastic number of bidders.

[36] Hendricks, Pinkse, and Porter (1999), for example, assume that bidders have the same beliefs about the number of competitors in all auctions at which the number of participating bidders falls in a certain range.

Taking $i = 1$, the inequality (8) and strict affiliation imply that for $\hat{\eta} > \eta$

$$\begin{aligned} \hat{b}(x_1; \hat{\eta}) &\equiv E_n \left[E[U_1 | X_1 = X_2 = x_1, X_j \leq x_1, \ j = 3, \ldots, n] \,|\hat{\eta}\right] \\ &< E_n \left[E[U_1 | X_1 = X_2 = x_1, X_j \leq x_1, \ j = 3, \ldots, n] \,|\eta\right] \\ &= \hat{b}(x_1; \eta). \end{aligned}$$

Since

$$\Pr(B^{(n-2:n)} \leq b) = \int_{-\infty}^{\infty} \Pr(\hat{b}(X^{(n-2:n)}; \eta) \leq b)\, d\pi(\eta|n)$$

strict affiliation implies

$$\Pr(B^{(n-2:n)} \leq b) > \int_{-\infty}^{\infty} \Pr(\hat{b}(X^{(n-2:n)}; \eta) \leq b) d\pi(\eta|n-1). \tag{14}$$

Similarly,

$$\Pr(B^{(n-1:n)} \leq b) > \int_{-\infty}^{\infty} \Pr(\hat{b}(X^{(n-1:n)}; \eta) \leq b)\, d\pi(\eta|n-1). \tag{15}$$

Using (4) and the fact that $\hat{b}(\cdot; \eta)$ is strictly increasing, we obtain the testable stochastic dominance relation

$$\begin{aligned} \Pr(B^{(n-2:n-1)} \leq b) &= \int_{-\infty}^{\infty} \Pr(\hat{b}(X^{(n-2:n-1)}; \eta) \leq b) d\pi(\eta|n-1) \\ &= \int_{-\infty}^{\infty} \left[\frac{2}{n} \Pr(\hat{b}(X^{(n-2:n)}; \eta) \leq b) + \frac{n-2}{n} \Pr(\hat{b}(X^{(n-1:n)}; \eta) \leq b) \right] d\pi(\eta|n-1) \\ &< \frac{2}{n} \Pr(B^{(n-2:n)} \leq b) + \frac{n-2}{n} \Pr(B^{(n-1:n)} \leq b) \end{aligned}$$

where the inequality follows from (14) and (15). □

6.1.2 First-Price Auctions

Bidder uncertainty over the number of opponents is a potentially more difficult problem in the case of a first-price auction, since bidding strategies will depend on η, even in a PV auction. With private values, after observing signal η each bidder i solves

$$\max_b \quad (u_i - b) \Pr(\max_{j \neq i} B_j \leq b | U_i = u_i, \eta)$$

giving first-order condition

$$b_i + \frac{\Pr(\max_{j \neq i} B_j \leq b_i | B_i = b_i, \eta)}{\frac{\partial}{\partial x} \Pr(\max_{j \neq i} B_j \leq x | B_i = b_i, \eta)\big|_{x = b_i}} = u_i. \tag{16}$$

If the econometrician observes a set of auctions in which η is fixed, this relation between bids and valuations can be used in essentially the same way that (12) was used above. For example, in the symmetric IPV case, observation of the winning bid in auctions with fixed η is still sufficient to identify $F_{\mathbf{U}}(\cdot)$. Let $\tilde{\pi}(n|\eta)$ denote the probability of there being n bidders when signal η is observed. Fixing η and letting B^{win} denote the winning bid, we observe $\Pr(B^{win} \leq b|\eta)$, which is equal to

$$\sum_{n=2}^{\infty} \tilde{\pi}(n|\eta) \Pr(B^{(n:n)} \leq b|\eta) = \sum_{n=2}^{\infty} \tilde{\pi}(n|\eta) \Pr(B \leq b|\eta)^n \tag{17}$$

Since (17) strictly increases in $\Pr(B_i \leq b|\eta)$ and $\tilde{\pi}(n|\eta)$ is observed directly, $\Pr(B_i \leq b|\eta)$ is identified. This enables construction of $\Pr(\max_{j \neq i} B_j \leq b_i|\eta)$, giving (through (16)) identification of $U^{(n:n)}$ for each n such that $\tilde{\pi}(n|\eta) > 0$. From the distribution of $U^{(n:n)}$, (3) can be used to identify $F_U(\cdot)$, which completely characterizes $F_{\mathbf{U}}(\cdot)$. $\square$

Our identification results for other private values models of first-price auctions can be extended in similar fashion. Testing of the PV hypothesis can be achieved by comparing distributions of $U^{(j:n)}$ for different values of η: under the PV hypothesis, these distributions will be identical; with common values we recover the distribution of $E[U_i|X^{(j:n)} = \max_{k \neq i} X_k = x_i, \eta]$ rather than that of $U^{(j:n)}$. This distribution in auctions where signal $\eta = \eta_1$ is observed will first-order stochastically dominate that in auctions where signal $\eta = \eta_2 > \eta_1$ is observed.

6.2 Reserve Prices

In many auctions the seller announces a reserve price for the auction, which can be viewed as an initial bid entered by the seller. For simplicity we have assumed reserve price lies below the support of bidders' valuations. When the reserve price r_0 is in the interior of the support, with positive probability some potential bidders will be unwilling to submit a bid. This creates a discrepancy between the number of potential bidders, N, and the number of participating bidders, n. However, by conditioning on the participation threshold, many of our results can be extended.[37]

Corollary 4 *In the IPV model of first-price, second-price, or ascending auctions, suppose that the reserve price, r_0, is fixed and that $F_U(r_0) \in (0,1)$.*
(i) If only one bid from each auction is observed, the distribution of U_i conditional on $U_i > r_0$, denoted $F_U(\cdot|r_0) = \frac{F_U(\cdot)-F_U(r_0)}{1-F_U(r_0)}$, is identified on $[r_0, \infty]$.
(ii) If either (a) two bids from each auction are observed or (b) a single bid is observed in auctions

[37] For first-price auctions, Bajari and Hortaçsu (2000) use a parametric model to simultaneously estimate a model of entry and bidding. Guerre, Perrigne, and Vuong (2000) includes a discussion of binding reserve prices in the IPV model. See also the recent paper by Li (2000). Hendricks, Pinkse and Porter (1999) estimate a model of stochastic participation and reserve prices in common value auctions.

with different numbers of participating bidders, the IPV model is testable.

(iii) If N is fixed and the number of participating bidders is observed, N and $F_U(r_0)$ are identified.

Proof. Because each potential bidder i participates when $x_i > r_0$, the common distribution of the participating bidders' valuations is $F_U(\cdot|r_0)$. Parts (i) and (ii) then follow from Propositions 1 and 16 and the remarks regarding testing following the proof of Proposition 16. Because the participation rule for each potential bidder is binomial with parameter $\lambda = F_U(r_0)$, both N and $F_U(r_0)$ are identified from the distribution of n when N is fixed. □

Similar extensions can be made for identification of the other models considered above. The following result shows that our tests of the PV and CV models can also be applied in this case.[38]

Corollary 5 *In a first-price, second-price, or ascending auction, suppose the reserve price, r_0, is fixed and in the interior of the support of each U_i. Both the PV model and the CV model are testable if we observe bids $B^{(j:n)}$ and $B^{(j+1:n)}$ (with $2 \leq j < n$) in all n-bidder auctions and bids $B^{(j:n-1)}$ in all $(n-1)$-bidder auctions.*

Proof. Participation is determined as in Milgrom and Weber (1982). Let x_0 denote the lowest type to participate in equilibrium. Participating bidders draw their types $X_1, \ldots, X_n$ from the distribution $F_{\mathbf{X}}(\cdot)$ truncated at $(x_0, \ldots, x_0)$. Because exchangeability is preserved by this truncation, the recurrence relation (4) still holds under the PV hypothesis, while the inequalities (9), (10), and (13) hold under the CV hypothesis. □

7 Conclusion

This paper has established new identification results for standard auction models. For second-price and ascending auctions, our main results can be summarized as follows (where the results apply for symmetric bidders unless otherwise noted):

1. The independent private values model is identified from the transaction price, and it is testable if more than one bid is observed in each auction.

2. In a second-price auction, observing the winner's identity and bid identifies the asymmetric independent private values model. In an ascending auction, this model is identified if we observe either the lowest bid and bidder identity or the bids of the top two losing bidders and the identities of the top three bidders.

[38] With a binding reserve price, one can also test the PV hypothesis $\zeta(x_0; m) = r_0$ against the alternative $\zeta(x_0; m) > r_0$ implied by the CV model (Milgrom and Weber (1982)). This testing approach has been proposed for first-price auctions by Hendricks, Pinkse, and Porter (1999).

3. In second-price auctions, but not ascending auctions, the "conditionally independent private values" model—where bidders' private values are the sum of a common component and an idiosyncratic component, and idiosyncratic components are independent conditional on the common component—is identified and testable if in addition, the common and idiosyncratic components are independent, and if all bids are observed in each auction. This extends to asymmetric bidders if identities are observed as well.

4. If the common component of values is observed ex post, the conditionally independent private values model is identified from the transaction price, and if two bids are observed, it is testable.

5. The unrestricted private values model is not identified from incomplete sets of bids.

6. Even with potentially asymmetric bidders, the unrestricted private values model is identified from the transaction price (and testable when two bids are observed) if there are bidder-specific covariates with sufficient variation.

7. The private values model is testable against the common values alternative, and vice-versa, if the transaction price and the next-highest bid are observed in two auction data sets with n and $n-1$ bidders, respectively (and the variation in participation is exogenous). This holds even though these assumptions do not guarantee identification.

8. In second-price auctions, but not ascending auctions, the "linear mineral rights model"—where bidders' expected values are linear in bidder signals, signals are linear in the common value and an idiosyncratic component, and idiosyncratic components are independent conditional on the common component—is identified and testable if, in addition, the common value is independent of the idiosyncratic component, and all bids are observed in each auction. This extends to asymmetric bidders if identities are observed as well.

9. In a second-price auction, if the common value is observed ex post, the linear mineral rights model is identified from the transaction price. In an ascending auction, we must observe the lowest bid rather than the transaction price.

Now consider how these results extend to first-price auctions with affiliated signals. Result (1) was established for first-price auctions by Guerre, Perrigne, and Vuong (1995), and results (3) and (8) were established by Li, Perrigne, and Vuong (1999). In this paper, we show that results (2) and (5) extend directly to first-price auctions, while results (4), (6), (7), and (9) extend if the top two bids in each auction are observed.

Most of the prior literature on structural estimation in ascending and second-price auctions has focused on parametric identification strategies. For first-price auctions, the prior literature

provides nonparametric identification results only when all bids are observed, or in the symmetric IPV model. Our results both indicate environments in which weaker identifying assumptions can be employed and suggest cases in which identification can only be obtained via functional form assumptions. Our results further suggest strategies for the design of new data sets. Increasingly, researchers are able to collect different elements of auction data (typically at some cost); this might be the case when obtaining data from internet auctions, as the researcher might choose whether to record the entire history of an auction as opposed to just information about the total number of participants and the transaction price. Our results suggest that in many environments, just the top two or three bids provide sufficient information for identification and/or testing.

This paper has focused primarily on identification. In general, identification is necessary but not sufficient for existence of a consistent estimator. Many of our identification proofs suggest estimation strategies, and for many of the identified models we have pointed to nonparametric estimation approaches that may be applicable. In most of these cases consistency is transparent; however, we have left the derivation of the asymptotic properties of these and other potential estimators for future work, along with their application to bidding data.

References

Arnold, Barry C., N. Balakrishnan, and H.N. Nagaraja (1992). *A First Course in Order Statistics.* New York: John Wiley & Sons.

Athey, Susan (2000). "Single Crossing Properties and the Existence of Pure Strategy Equilibria in Games of Incomplete Information," *Econometrica*, forthcoming.

Athey, Susan and Jonathan Levin (2000). "Information and Competition in U.S. Forest Service Timber Auctions," *Journal of Political Economy*, forthcoming.

Bajari, Patrick (1998). "Econometrics of the First Price Auction with Asymmetric Bidders," working paper, Stanford.

Bajari, Patrick and Ali Hortaçsu (2000). "Winner's Curse, Reserve Prices and Endogenous Entry: Empirical Insights from eBay Auctions," working paper, Stanford University.

Balasubramanian, K. and N. Balakrishnan (1994). "Equivalence of Relations for Order Statistics for Exchangeable and Arbitrary Cases," *Statistics and Probability Letters*, 21, 405–407.

Baldwin, Laura, Robert Marshall, and Jean-Francois Richard, 1997, "Bidder Collusion in U.S. Forest Service Timber Sales," *Journal of Political Economy* 105 (4), 657–99.

Berman, Simon M. (1963). "Note on Extreme Values, Competing Risks, and Semi-Markov Processes, *Annals of Mathematical Statistics*, 34, 1104–1106.

Campo, Sandra, Isabelle Perrigne and Quang Vuong (2000). "Asymmetry in First-Price Auctions with Affiliated Private Values," working paper, USC.

Chow, Yuan Shih and Henry Teicher (1997). *Probability Theory: Independence, Interchangeability, Martingales,* New York: Springer.

Cox, David R. (1959). "The Analysis of Exponentially Distributed Lifetimes with Two Types of Failures," *Proceedings of the Royal Statistical Society, B,* 148, 82-117.

Crandall, Robert W. (1998). "New Zealand Spectrum Policy: A Model for the United States?" *Journal of Law and Economics*, 41, 821–40.

David, Herbert A. (1981). *Order Statistics*, 2nd edition, New York: John Wiley & Sons.

Deltas, George and Idranil Chakraborty (1997). "A Two-Stage Approach to Structural Econometric Analysis of First-Price Auctions," working paper, University of Illinois.

Donald, Stephen G., Harry J. Paarsch, and Jacques Robert (1999). "Identification, Estimation, and Testing in Empirical Models of Sequential, Ascending-Price Auctions with Multi-Unit Demand: An Application to Siberian Timber-Export Permits," working paper, University of Iowa.

Donald, Stephen G. and Harry J. Paarsch (1996). "Identification, Estimation, and Testing in Parametric Empirical Models of Auctions within the Independent Private Values Paradigm," *Econometric Theory*, 12, 517–567.

Elyakime, Bernard, Jean–Jacques Laffont, Patrice Loisel, and Quang Vuong (1994). "First-Price Sealed-Bid Auctions with Secret Reserve Prices," *Annales d'Economie et Statistiques*, 34: 115–141.

Engelbrecht-Wiggans, Richard, John List, and David Lucking-Reiley (1999). "Demand Reduction in Multi-unit Auctions with Varying Numbers of Bidders: Theory and Field Experiments," working paper, Vanderbilt University.

Guerre, Emmanuel, Isabelle Perrigne and Quang Vuong (1995). "Nonparametric Estimation of First-Price Auctions," Working Paper, Economie et Sociologie Rurales, Toulouse, no. 95-14D.

Guerre, Emmanuel, Isabelle Perrigne and Quang Vuong (2000). "Optimal Nonparametric Estimation of First-Price Auctions," *Econometrica*, 68, 525–574.

Haile, Philip A. (2000). "Auctions with Resale Markets: An Application to U.S. Forest Service Timber Sales," *American Economic Review*, forthcoming.

Haile, Philip A. and Elie T. Tamer (2000). "Inference with an Incomplete Model of English Auctions," working paper, University of Wisconsin-Madison.

Haile, Philip A., Han Hong, and Matthew Shum (2000). "Nonparametric Tests for Common Values in First-Price Auctions," working paper, University of Wisconsin-Madison.

Han, Aaron and Jerry A. Hausman (1990). "Flexible Parametric Estimation of Duration and Competing Risk Models," *Journal of Applied Econometrics*, 5, 1–28.

Heckman, James J. and Bo E. Honoré (1989). "The Identifiability of the Competing Risks Model," *Biometrika*, 76, 325-330.

Heckman, James J. and Bo E. Honoré (1990). "The Empirical Content of the Roy Model," *Econometrica*, 58, 1121–1149.

Hendricks, Kenneth and Harry J. Paarsch (1995) "A Survey of Recent Empirical Work Concerning Auctions," *Canadian Journal of Economics*, 28, 403–426.

Hendricks, Kenneth, Joris Pinkse and Robert H. Porter (1999). "Empirical Implications of Equilibrium Bidding in First-Price, Symmetric, Common Value Auctions," working paper, Northwestern University.

Hendricks, Kenneth and Robert H. Porter (1988). "An Empirical Study of an Auction with Asymmetric Information," *American Economic Review*, 78, 865–883.

Hendricks, Kenneth and Robert H. Porter (2000). "Lectures on Auctions: An Empirical Perspective," working paper, Northwestern University

Hong, Han and Matthew Shum (1999). "Econometric Models of Ascending Auctions," working paper, Princeton University.

Kotlarski, Ignacy (1966). "On Some Characterization of Probability Distributions in Hilbert Spaces," *Annali di Matematica Pura ed Applicata*, 74, 129–134.

Laffont, Jean–Jacques (1997). "Game Theory and Empirical Economics: The Case of Auction Data," *European Economic Review*, 41, 1–35.

Laffont, Jean-Jacques, Hervé Ossard and Quang Vuong (1995). "Econometrics of First-Price Auctions," *Econometrica*, 63, 953–980.

Laffont, Jean-Jacques and Quang Vuong (1993). "Structural Econometric Analysis of Descending Auctions," *European Economic Review*, 37, 329–341.

Laffont, Jean-Jacques and Quang Vuong (1996). "Structural Analysis of Auction Data," *American Economic Review, Papers and Proceedings*, 86, pp. 414–420.

Li, Tong, (2000), "Econometrics of First-Price Auctions with Binding Reservation Prices," Mimeo, Indiana University.

Li, Tong, Isabelle Perrigne and Quang Vuong (1998). "Structural Estimation of the Affiliated private values Model with an Application to OCS Auctions," Mimeo, U.S.C.

Li, Tong, Isabelle Perrigne and Quang Vuong (1999). "Conditionally Independent Private Information in OCS Wildcat Auctions," Mimeo, U.S.C.

Li, Tong and Quang Vuong (1998). "Nonparametric Estimation of the Measurement Error Model Using Multiple Indicators." *Journal of Multivariate Analysis,* 65, 139-165.

Lucking-Reiley, David (2000). "Auctions on the Internet: What's Being Auctioned, and How?" *Journal of Industrial Economics*, forthcoming.

Matthews, Steven (1987). "Comparing Auctions for Risk Averse Buyers: A Buyer's Point of View," *Econometrica*, 55, 633–646.

McAfee, R. Preston and John McMillan (1987a): "Auctions and Bidding," *Journal of Economic Literature*, 25, 669–738.

McAfee, R. Preston and John McMillan (1987b). "Auctions with a Stochastic Number of Bidders," *Journal of Economic Theory*, 43, 1–19.

McAfee, R. Preston and John McMillan (1996). "Analyzing the Airwaves Auction," *Journal of Economic Perspectives*,10, 159–175.

McAfee, R. Preston, Wendy Takas, and Daniel R. Vincent (1999). "Tariffying Auctions," *RAND Journal of Economics*, 30, 158–179.

McFadden, Daniel (1989). "Testing for Stochastic Dominance," in *Studies in the Economics of Uncertainty: In Honor of Josef Hadar*, New York: Springer.

Milgrom, Paul R. (1985): "The Economics of Competitive Bidding: A Selective Survey," in L. Hurwicz, D. Schmeidler, and H. Sonnenschein, eds., *Social Goals and Social Organization: Essays in Memory of Elisha Pazner*, Cambridge: Cambridge University Press, 261–289.

Milgrom, Paul R. and Robert J. Weber (1982). "A Theory of Auctions and Competitive Bidding," *Econometrica*, 50, 1089–1122.

Paarsch, Harry J. (1992a). "Deciding Between Common and Private Values Paradigms in Empirical Models of Auctions," *Journal of Econometrics.*

Paarsch, Harry J. (1992b). "Empirical Models of Auctions and an Application to British Columbian Timber Sales," Research Report 9212, University of Western Ontario.

Perrigne, Isabelle and Quang Vuong (1999). "Structural Econometrics of First-Price Auctions: A Survey," *Canadian Journal of Agricultural Economics*, forthcoming.

Prakasa Rao, B.L.S. (1992). *Identifiability in Stochastic Models: Characterization of Probability Distributions,* San Diego: Academic Press .

Romano, Joseph (1988). "A Bootstrap Revival of Some Nonparametric Distance Tests," *Journal of the American Statistical Association*, 83, pp. 698–708.

Shaked, Moshe (1979). "Some Concepts of Positive Dependence for Bivariate Interchangeable Distributions," *Annals of the Institute of Statistical Mathematics*, 31, pp. 67–84.

Tsiatis, Anastasios A. (1975). "A Nonidentifiability Aspect of the Problem of Competing Risks," *Proceedings of the National Academy of Sciences, USA*, 72, 20-22.

Wilson, Robert (1992): "Strategic Analysis of Auctions," in R. J. Aumann and S. Hart, eds., *Handbook of Game Theory, Volume 1*, Amsterdam: North-Holland.

Wilson, Robert (1998): "Sequential Equilibria of Asymmetric Ascending Auctions: The Case of Lognormal Distributions," *Economic Theory,* 12, 433–440.

Date Due

Lib-26-67